System Concepts

COSC 5060

Dr. Ephraim Martin

Mark Zieg

1st Class Session

August 13, 2001

Read chapter 1 ("Formal Logic") and complete the following problems to hand in:

p30 (§1.2): 1, 4, 6, 12
p41 (§1.3): 4a, 8b, 9a, 12a, 15c, 16a
p56 (§1.4): 2, 4, 9, 17

§1.2 Propositional Logic

1. Justify each step in the proof sequence of

$$[A \to (B \vee C)] \wedge B' \wedge C' \to A'$$

1.	$A \to (B \vee C)$	hyp
2.	B'	hyp
3.	C'	hyp
4.	$B' \wedge C'$	con, 2, 3
5.	$(B \vee C)'$	DeMorgan, 4
6.	A'	mp, 1, 5

4. Use propositional logic to prove that the argument is valid:

$$(A \to B) \wedge [A \to (B \to C)] \to (A \to C)$$

1.	$A \to B$	hyp
2.	$A \to (B \to C)$	hyp
3.	A	hyp
4.	$B \to C$	mp, 3, 2
5.	B	mp, 1, 3
6.	C	mp, 4, 5

6. Use propositional logic to prove that the argument is valid:

$$A' \wedge (A \vee B) \to B$$

1.	A'	hyp
2.	$A \vee B$	hyp
3.	$A' \to B$	imp, 2
4.	B	mp, 1, 3

12. Use propositional logic to prove that the argument is valid:

$$(P \vee Q) \wedge P' \to Q$$

1.	$P \vee Q$	hyp
2.	P'	hyp
3.	$P' \to Q$	imp, 1
4.	Q	mp, 2, 3

§1.3 Quantifiers, Predicates, and Validity

4a. Find an interpretation in which it is true, and one in which it is false:

$(\forall x)\ (\ [A(x) \vee B(x)\ ] \wedge [A(x) \wedge B(x)]'\ )$

Domain = all dinosaurs

True: $A(x)$ = dinosaur had two legs
$B(x)$ = dinosaur had four legs

All dinosaurs had two legs *or* four legs, and *no* dinosaurs had *both* four legs *and* two legs.

False: $A(x)$ = dinosaur eats meat
$B(x)$ = dinosaur eats plants

All dinosaurs ate meat *or* ate plants, and *no* dinosaurs ate *both* meat *and* plants. (Ornitholestes was believed to be an omnivore.)

8b. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff.

$J(x)$ is "x is a judge."
$L(x)$ is "x is a lawyer."
$W(x)$ is "x is a woman."
$C(x)$ is "x is a chemist."
$A(x,y)$ is "x admires y."

"No woman is both a laywer and a chemist."

$$[\ (\exists x)(W(x) \wedge L(x) \wedge C(x))\ ]'$$

9a. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff.

$C(x)$ is "x is a Corvette."
$F(x)$ is "x is a Ferarri."
$P(x)$ is "x is a Porche."
$S(x,y)$ is "x is slower than y."

"Nothing is both a Corvette and a Ferrari."

$$(\exists x)(C(x) \rightarrow F(x)')$$

12a. Give English language translations of the following wffs if

$L(x,y)$ is "x loves y."
$M(x)$ is "x is a man."
$P(x)$ is "x is pretty."
$W(x)$ is "x is a woman."
$H(x)$ is "x is handsome."
j is "John."
k is "Kathy."

$H(j) \wedge L(k,j)$

"John is handsome and is loved by Kathy." (Whether or not the love is unrequited is not addressed by the wff.)

15c. Explain why each wff is valid:

$$(\exists x)\ (\forall y)\ P(x,y) \rightarrow (\forall y)\ (\exists x)\ P(x,y)$$

Since the existence of the *x* was stated prior to and unpredicated on any qualification of *y*, *x*'s existence may be taken as unrelated to the scope of *y*, up to and including universal scope. *Moreover*, the antecedent of the implication states as hypothesis that the known *x* has the relationship $P(x,y)$ for every possible *y*. Since the scope of *y* is universal, then it is a tautology that the same *x*, whose existence has already been hypothesized, will therefore still exist in the same relationship $P(x,y)$ for every *y*.

With regards to last week's class discussion, here is my reasoning for believing that the converse does not necessarily hold true, ie:

$$(\forall y)\ (\exists x)\ P(x,y) \not\rightarrow (\exists x)\ (\forall y)\ P(x,y)$$

The domain is *Gilligan's Island*, which featured three women (Ginger, Mary Anne, and Mrs. Howe) and four men (Gilligan, Skipper, Professor, and Mr. Howe).

Let *x* = "a man."
Let *y* = "a woman."
Let $P(x,y)$ = "Man *x* has a lifelong and monogomous relationship with woman *y*."

$(\forall y)\ (\exists x)\ P(x,y)$ translates as "For every woman on *Gilligan's Island*, there exists a man on the island with whom she has a monogomous and lifelong relationship."
$(\exists x)\ (\forall y)\ P(x,y)$ translates as "There is a man on *Gilligan's Island* who has a monogonous and lifelong relationship with every woman on the island."

Clearly, $(\forall y)\ (\exists x)\ P(x,y)$ could be true (personally, I'd match up Gilligan with Mary Anne and Ginger with the Professor ☺), yet would certainly not imply $(\exists x)\ (\forall y)\ P(x,y)$ – which is impossible anyway.

To take another example from pop culture (this time, a song from the 80's rock group *Poison*) the bromide "every rose has its thorn" does not imply "there is a rose for every thorn" – some bristles are found lurking in thorny math problems!

16a. Give interpretations to prove that each of the following wffs is not valid:

$$(\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)[\ A(x) \wedge B(x)]$$

The domain is Isla Sorna, ie *Jurassic Park* "Site B."
$A(x)$ = Dinosaur *x* has a skull with 12" of solid bone.
$B(x)$ = Dinosaur *x* has 6" dagger-like teeth.

The wff is not valid according to *Jurassic Park: The Lost World* because, although Isla Sorna was populated by pachycephalosauruses ($A(x)$) and Tyranosaurus Rexen ($B(x)$), there were no mutant bone-domed carnivores.

§1.4 Predicate Logic

2. Consider the wff

$$(\forall x)\,[(\exists y)P(x,y) \wedge (\exists y)\,Q(x,y)] \rightarrow (\forall x)\,(\exists y)\,[P(x,y) \wedge Q(x,y)]$$

a. Find an interpretation to prove that this wff is not valid.

Domain: All possible implications $A \rightarrow$ B
$P(x,y)$ = "y is the converse of x"
$Q(x,y)$ = "y is the contrapositive of x"

The wff would therefore translate as, "For all wffs x of the form A $\rightarrow$ B, there exists a wff y which is the converse of x, and a wff y which is the contrapositive of x. This implies that for all wffs x, there exists a wff y which is both the converse and the contradiction of x."

This is provably false, since for any wff x of the form A $\rightarrow$ B, the contrapositive will have the form B' $\rightarrow$ A', while the converse will have the form B $\rightarrow$ A. These are clearly different (and non-equivalent) wffs.

We can easily show that that B'$\rightarrow$ A' is unequivalent to B $\rightarrow$ A by counterexample, by showing a case where the latter is true yet the former is false:

If B is "being a book" and A is "all things made of paper", then we know that B $\rightarrow$ A is true (if a thing is a book, then the thing is made of paper). However, just because a thing is *not* a book does not mean that a thing is not made of paper..

b. Find the flaw in the following "proof" of this wff.

1.	$(\forall x)\,[(\exists y)P(x,y) \wedge (\exists y)\,Q(x,y)]$	hyp
2.	$(\forall x)\,[P(x,a) \wedge Q(x,a)]$	ei, 1
3.	$(\forall x)\,(\exists y)\,[P(x,y) \wedge Q(x,y)]$	eg, 2

In step (2), the same value a is instantiated in two different parts of the wff. There is no reason to believe, based on the hypothesis given, that the same value a for variable y would necessarily hold true for both $P(x,y)$ and $Q(x,y)$. Therefore, this should have been done in two Existential Instantiations, one using the value a and the other using the value b. This would have prevented the incorrect Existential Generalization transformation applied in step (3).

4. Prove that the wff is a valid argument:

$$(\forall x)\,P(x) \rightarrow (\forall x)\,[P(x) \vee Q(x)]$$

1.	$(\forall x)\,P(x)$	hyp
2.	$P(x)$	ui, 1
3.	$P(x) \vee Q(x)$	add, 2
4.	$(\forall x)\,[P(x) \vee Q(x)]$	ug, 3

9. Either prove that the wff is a valid argument or give an interpretation in which it is false:

$$(\exists x)\,[A(x) \wedge B(x)] \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$$

1.	$(\exists x)\,[A(x) \wedge B(x)]$	hyp
2.	$A(x) \wedge B(x)$	ei, 1
3.	$A(x)$	sim, 2
4.	$B(x)$	sim, 2
5.	$(\exists x)\,A(x)$	eg, 3
6.	$(\exists x)\,B(x)$	eg, 4
7.	$(\exists x)A(x) \wedge (\exists x)B(x)$	con, 5, 6

17. Either prove that the wff is a valid argument or give an interpretation in which it is false:

$$[P(x) \rightarrow (\exists y)Q(x,y)] \rightarrow (\exists y)[P(x) \rightarrow Q(x,y)]$$

1.	$P(x) \rightarrow (\exists y)Q(x,y)$	hyp
2.	$P(x)$	temp hyp
3.	$\quad(\exists y)Q(x,y)$	imp, 1, 2
4.	$\quad Q(x,a)$	ei, 3
5.	$P(x) \rightarrow Q(x,a)$	temp hyp discharged
6.	$(\exists y)[P(x) \rightarrow Q(x,y)]$	eg, 5

System Concepts

COSC 5060

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2nd Class Session

August 20, 2001

Read chapter 2 ("Proofs, Recursion, and Analysis of Algorithms")
and complete the following problems to hand in:

p15 (§1.1): 11c, 15d, 19d
p30 (§1.2): 8, 25, 29
p41 (§1.3): 4d, 9e, 15e
p56 (§1.4): 10, 18, 24
p76 (§1.6): 3, 8
p93 (§2.1): 14, 22

§1.4 Statements, Symbolic Representation, and Tautologies

11c. Construct truth tables for the following wffs. Note any tautologies or contradictions.

$A \wedge (A' \vee B')'$

First, let's simplify the wff, to minimize the number of columns needed for the truth table:

1.	$A \wedge (A' \vee B')'$	hyp
2.	A	sim, 1
3.	$(A' \vee B')'$	sim, 1
4.	$A \wedge B$	DeMorgan, 3
5.	B	sim, 4
6.	$A \wedge B$	con, 2, 5

A	*B*	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

15d. Verify by constructing truth tables that the follow wffs are tautologies.

$A \rightarrow (A \vee B)$

A	*B*	$A \vee B$	$A \rightarrow A \vee B$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

19d. Use algorithm *TautologyTest* to prove that the following are tautologies.

$(A \wedge B) \wedge B' \rightarrow A$

Comment: On the surface, this appears true by inconsistency. $B \wedge B'$ is a contradiction, so the wff essentially says nothing at all about the value of A – meaning that the implication is true, ie that, *if* A, *then* A, *else* not A.

1.	Consider $(A \wedge B) \wedge B' \rightarrow A$ to be of the form $P \rightarrow Q$	
2.	Assume P is true.	
3.	Assume Q is false.	
4.	Assume $(A \wedge B) \wedge B'$ is true	(1, 3)
5.	Assume A is false	(1, 3)
6.	Assume $(A \wedge B)$ is true	(4, sim)
7.	Assume B' is true	(4, sim)
8.	Assume A is true	(6, sim)
9.	Assume B is true	(6, sim)
10.	Assume B is false	(7)
11.	*** All letters now have truth letters***	
12.	Totals are:	
13.	A: false, true	(5, 9)
14.	B: true, false	(9, 10)
15.	<u>Both A and B end up with multiple truth values</u>	
16.	∴ contradiction, ie $(A \wedge B) \wedge B' \rightarrow A$ is a tautology	

§1.2 Propositional Logic

8. Use propositional logic to prove that the argument is valid:

$$(A' \to B') \wedge B \wedge (A \to C) \to C$$

1.	$A' \to B'$	hyp
2.	B	hyp
3.	$A \to C$	hyp
4.	$A \vee B'$	imp, 1
5.	$B' \vee A$	comm, 4
6.	$B \to A$	imp, 5
7.	A	mp, 6, 2
8.	C	mp, 3, 7

25. Use propositional logic to prove the argument valid:

$$(A \vee B) \wedge (A \to C) \wedge (B \to C) \to C$$

1.	$A \vee B$	hyp
2.	$A \to C$	hyp
3.	$B \to C$	hyp
4.	$A' \vee C$	imp, 2
5.	$B' \vee C$	imp, 3
6.	$(A' \vee C) \wedge (B' \vee C)$	con, 4, 5
7.	$(A' \wedge B') \vee C$	rev dist, 6?
8.	$(A' \wedge B')' \to C$	imp, 7
9.	$(A \vee B) \to C$	DeMorgan, 8
10.	C	mp, 9, 1

29. Using propositional logic, prove that each argument is valid. Use the statement letters shown.

If the program is efficient, it excutes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore, it has a bug. (E, Q, B)

$$(E \to Q) \wedge (E \vee B) \wedge Q' \to B$$

1.	$E \to Q$	hyp
2.	$E \vee B$	hyp
3.	Q'	hyp
4.	$E' \vee Q$	imp, 1
5.	E'	ds, 4, 3
6.	B	ds, 2, 5

§1.3 Quantifiers, Predicates, and Validity

4d. For each wff, find an interpretation in which it is true and one in which it is false.

$(\exists x)[\ A(x) \wedge (\forall y)\ B(x,y)\ ]$

True:

The domain is all positive even integers.
A(x) = "x < 3"
B(x,y) = "y / x is a positive integer"

False:

The domain is all positive even integers.
A(x) = "x > 3"
B(x,y) = "y / x is a positive integer"

9e. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff.

$C(x)$ is "x is a Corvette."
$F(x)$ is "x is a Ferrari."
$P(x)$ is "x is a Porche."
$S(x,y)$ is "x is slower than y."

"Some Porches are slower than no Corvette."

$$(\exists x)\ (\forall y)\ [(P(x) \wedge C(y)) \rightarrow S(x,y)']$$

15e. Explain why each wff is valid:

$$(\forall x)[A(x) \rightarrow B(x)] \rightarrow [(\forall x)A(x) \rightarrow (\forall x)B(x)]$$

This is valid because in both sides of the primary implication, the predicate $A(x)$ filters the scope of x which is then evaluated by $B(x)$.

In the antecedent, $(\forall x)$ starts the wff by declaring x to be "wide open" – anything within the domain. However, the predicate $A(x)$ immediately narrows down the possible values of x to only those matching $A(x)$. It is this subset of $\forall x$ which is then implied to be also true under $B(x)$.

In the consequent wff, this is not immediately obvious, because we now see the notation $(\forall x)B(x)$ and think, "Whoa, we don't know that to be true – we don't know that $B(x)$ is true for *all* x, because that wasn't said in the antecedent wff – we only knew that $B(x)$ was true if $A(x)$."

However, the same reasoning holds true, because the antecedent in the consequent wff does apply a functionally similar filter on the input to $B(x)$. In this case, it states, essentially, that "For all x such that $A(x)$, then $B(x)$." Ie, we no longer start with a truly "global" x as we did in the first wff – we start with a pre-filtered subset of global, matching $A(x)$, which is exactly the antecedent we need to know that $B(x)$ is true for all filtered-possible values of x.

§1.4 Predicate Logic

10. Either prove that the wff is a valid argument or give an interpretation in which it is false.

$$(\exists x)\,[R(x) \vee S(x)] \rightarrow (\exists x)R(x) \vee (\exists x)S(x)$$

Start by re-writing the conclusion as:

$$[(\exists x)R(x)]' \rightarrow (\exists x)S(x)$$

1.	$(\exists x)\,[R(x) \vee S(x)]$	hyp
2.	$[(\exists x)R(x)]'$	hyp
3.	$(\forall x)R(x)'$	neg, 2
4.	$R(a)'$	ui, 3
5.	$R(a) \vee S(a)$	ei, 1
6.	$R(a)' \rightarrow S(a)$	imp, 5
7.	$S(a)$	mp, 5, 4
8.	$(\exists x)S(x)$	eg, 7

18. Either prove that the wff is a valid argument or give an interpretation in which it is false.

$$\exists x[P(x) \rightarrow Q(x)] \wedge \forall y[Q(y) \rightarrow R(y)] \wedge \forall x P(x) \rightarrow \exists x R(x)$$

1.	$\exists x[P(x) \rightarrow Q(x)]$	hyp
2.	$\forall y[Q(y) \rightarrow R(y)]$	hyp
3.	$\forall x P(x)$	hyp
4.	$P(a) \rightarrow Q(a)$	ei, 1
5.	$P(a)$	ui, 3
6.	$Q(a)$	mp, 4, 5
7.	$\exists x Q(x)$	eg, 6
8.	$Q(b) \rightarrow R(b)$	ui, 2
9.	$Q(b)$	ei, 7
10.	$R(b)$	mp, 8, 9
11.	$\exists x R(x)$	eg, 10

24. Using predicate logic, prove that each argument is valid. Use the predicate symbols shown.

Every computer science student works harder than somebody, and everyone who works harder than any other person gets less sleep than that prson. Maria is a computer science student. Therefore, Maria gets less sleep than someone else.
$C(x)$, $W(x,y)$, $S(x,y)$, m

$$\forall x[C(x) \rightarrow W(x,y)] \wedge \forall x[W(x,y) \rightarrow S(x,y)] \wedge C(m) \rightarrow \exists y S(m,y)$$

1.	$\forall x[C(x) \rightarrow W(x,y)]$	hyp
2.	$\forall x[W(x,y) \rightarrow S(x,y)]$	hyp
3.	$C(m) \rightarrow W(m,y)$	ui, 1
4.	$C(m)$	hyp
5.	$W(m,y)$	mp, 3, 4
6.	$W(m,y) \rightarrow S(m,y)$	ui, 2
7.	$S(m,y)$	mp, 6, 5
8.	$\exists y S(m,y)$	eg, 7

§1.6 Proof of Correctness

3. Verify the correctness of the following program segment with the preconditon and postcondition shown.

$\{x = 1\}$
$y = x + 3$
$y = 2 * y$
$\{y = 8\}$

{Q}		$\{x = 1\}$	
	S_0		$y = x + 3$
$\{R_1\}$		$\{y = 4\}$	
	S_1		$y = 2 * y$
{R}		$\{y = 8\}$	

8. Verify the correctness of the following program segment with the preconditon and postcondition shown.

$\{x = 7\}$
if $x \leq 0$ **then**
$y = x$
else
$y = 2 * x$
end if
$\{y = 14\}$

{Q}	$\{x = 7\}$	
{B}	$\{x \leq 0\}$	
$\{P_1\}$		$y = x$
$\{P_2\}$		$y = 2 * x$
{R}	$\{y = 14\}$	

$\{Q \wedge B\}\ P_1\ \{R\}$
$\{x = 7 \text{ and } x \leq 0\}\ y = x\ \{y = 14\}$
$\{x = 7 \text{ and } x \leq 0\}\ y = 7\ \{y = 14\}$

true because $\{Q \wedge B\}'$

$\{Q \wedge B\}'\ P_2\ \{R\}$
$\{x = 7 \text{ and } x > 0\}\ y = 2 * x\ \{y = 14\}$
$\{x = 7 \text{ and } x > 0\}\ y = 14\ \{y = 14\}$

true by assignment rule

14. Prove the given statement.

For every integer *n*, the number

$$3(n^2 + 2n + 3) - 2n^2$$

is a perfect square.

In other words, prove the following predicate wff (domain is all integers):

$$\forall x \exists y[3(x^2 + 2x + 3) - 2x^2 = y^2]$$

Start by simplifying the equation:

1. $3(x^2 + 2x + 3) - 2x^2 = y^2$
2. $3x^2 + 6x + 9 - 2x^2 = y^2$
3. $x^2 + 6x + 9 = y^2$
4. $(x + 3)(x + 3) = y^2$
5. $(x + 3)^2 = y^2$

We can now see that the equation is intuitively tautological – obviously, any quantity squared will always result in a perfect square.

22. Prove the given statement.

The sum of three consecutive integers is divisible by 3.

In other words, prove the following predicate wff:

$$\forall x[Q(x) \rightarrow Q(P(x))]$$

$$P(x) = \frac{x + (x+1) + (x+2)}{3}$$

$Q(x)$ = "*x* is an integer"

Again, begin by simplifying the equation $P(x)$:

1. $\dfrac{x + (x+1) + (x+2)}{3}$
2. $\dfrac{x + x + 1 + x + 2}{3}$
3. $\dfrac{3x + 3}{3}$
4. $\dfrac{3(x+1)}{3}$
5. $3\left(\dfrac{(x+1)}{3}\right)$

Again, the simplified equation makes its truth obvious – clearly, any quantity multiplied by three will be evenly divisible by 3.

COSC 5060
Dr. Martin

3rd Class Session
August 27, 2001

Mark Zieg

all those things we cannot know
we dream, we hypothesize
maybe these are secrets shared by those
watching from the sky
if we are only members of the human race
no supernatural beings from a
supranatural place
if you can't solve the problem
come and tell me to my face
– Geddy Lee, *My Favorite Headache*, “The Angels’ Share”

Read chapter 3 and complete the following problems to hand in:

p93 (§2.1): 18, 24, 27
p106 (§2.2): 7, 22, 37
p118 (§2.3): 7, 9
p137 (§2.4): 8, 15, 33, 37
p176 (§3.1): 6a, 7b, 10k, 17, 35

§2.1

18. Prove the given statement:

If $x^2 + 2x - 3 = 0$ then $x \neq 2$.

Proof by contradiction:

1. $x^2 + 2x - 3 = 0$ hyp
2. $x = 2$ hyp
3. 2*2 + 2(2) – 3 = 0
4. 4 + 4 – 3 = 0
5. 5 = 0
6. Contradiction!
7. $\therefore\ x^2 + 2x - 3 = 0 \rightarrow x \neq 2$

24. Prove the given statement:

The difference of two consecutive [integral] cubes is odd.

Start by expressing the statement mathematically and simplifying:

1. $(x+1)^3 - x^3$
2. $(x+1)(x+1)^2 - x^3$
3. $(x+1)(x^2 + 2x + 1) - x^3$
4. $(x^3 + 2x^2 + x + x^2 + 2x + 1) - x^3$
5. $(x^3 + 3x^2 + 3x + 1) - x^3$
6. $(x^3 + 3x^2 + 3x + 1) - x^3$
7. $3x^2 + 3x + 1$

Now we only need to prove that $3x^2 + 3x + 1$ is odd for all integers *x*. By definition, if *x* is odd, then *x* – 1 is even. Therefore, we can instead attempt to prove that $3x^2 + 3x$ is even.

1. $3x^2 + 3x$
2. $3x(x+1)$
3. $3(x)(x+1)$

The above product will be even if *any* of the three factors are even. We know by definition that either *x* or *x* + 1 is guaranteed to be even, so the whole expression will be even.

Therefore the statement is true.

27. Prove the given statement:

For any two numbers x and y, $|xy| = |x||y|$.

$|x|$ is defined as x if $x \geq 0$, $(-1)(x)$ otherwise (or $[0 - x]$).

Prove for all four possible cases:

Case	x	y
1	neg	neg
2	neg	pos
3	pos	neg
4	pos	pos

Case 1:

1. $|xy| = |x|\ |y|$
2. $xy = (0-x)(0-y)$
3. $xy = xy$
4. true

Case 2:

1. $|xy| = |x|\ |y|$
2. $(0 - xy) = (0-x)y$
3. $-xy = -xy$
4. true

Case 3:

Identical to Case 2 with x *and* y *transposed.*

Case 4:

1. $|xy| = |x|\ |y|$
2. $xy = xy$
3. true

§2.2

7. Use mathematical induction to prove that the statement is true for every positive integer *n*.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

To prove this via the 1st principle of mathematical induction, we must complete the following sub-proofs:

1. prove P(1)
2. assuming P(k), prove P(k+1)

Direct proof of #1:

$$1^2 = \frac{1(1+1)(2+1)}{6}$$
$$1 = \frac{1(2)(3)}{6}$$
$$1 = \frac{6}{6}$$
$$1 = 1$$
true

Direct proof of #2:

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$
$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$k(k+1)(2k+1) + 6(k+1)^2 = (k+1)(k+2)(2k+3)$$
$$(k^2+k)(2k+1) + 6(k^2+2k+1) = (k+1)(2k^2+7k+6)$$
$$2k^3 + k^2 + 2k^2 + k + 6k^2 + 12k + 6 = 2k^3 + 7k^2 + 6k + 2k^2 + 7k + 6$$
$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$
$$1 = 1$$
true

22. An *arithmetic progression* (*arithmetic sequence*) is a sequence of terms where there is an initial term *a* and each succeeding term is obtained by adding a *common difference d* to the previous term. Prove the formula for the sum of the first *n* terms of an arithmetic sequence ($n \geq 1$):

$$a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = \frac{n}{2}[2a+(n-1)d]$$

To prove this via the 1st principle of mathematical induction, we must complete the following sub-proofs:

1. prove P(1)
2. assuming P(k), prove P(k+1)

Direct proof of #1:

$$a = \frac{1(2a+d(0)}{2}$$

$$a = \frac{2a}{2}$$

$$a = a$$

true

Direct proof of #2:

$$a + (a+d) + (a+2d) + \cdots + [a+d(k-1)] + [a+d(k)] = \frac{(k+1)[2a+d(k)]}{2}$$

$$\frac{k(2a+d(k-1)}{2} + (a+dk) = \frac{2ak+dk^2+2a+dk}{2}$$

$$k(2a+dk-d) + 2a + 2dk = 2ak + dk^2 + 2a + dk$$

$$dak + dk^2 - dk + 2a + 2dk = dk^2 + 2ak + dk + 2a$$

$$dk^2 + 2ak + dk + 2a = dk^2 + 2ak + dk + 2a$$

$$1 = 1$$

true

37. Prove that the statement is true for every positive integer.

$3^{2n} + 7$ is divisible by 8.

To prove this via the 1st principle of mathematical induction, we must complete the following sub-proofs:

1. prove P(1)
2. assuming P(k), prove P(k+1)

Direct proof of #1:

$$3^{2(0)} + 7 = 8x$$
$$3^{0} + 7 = 8x$$
$$1 + 7 = 8x$$
$$8 = 8x$$
$$1 = x$$
true

Direct proof of #2:

$$3^{2(k+1)} + 7 = 8y$$
$$3^{2k+2} + 7 = 8y$$
$$(3^{2k})(3^{2}) + 7 = 8y$$
$$(8x - 7)(9) + 7 = 8y$$
$$72x - 63 + 7 = 8y$$
$$72x - 56 = 8y$$
$$8(9x - 8) = 8y$$
true

§2.3

7. Use the Euclidean algorithm to find the greatest common divisor of the given numbers:

(1326, 252)

5	3	1	4	2
252)1326	66)252	54)66	12)54	6)12
1260	198	54	48	12
66	54	12	6	0

$GCD = 6$

9. Prove that the program segment is correct by finding and proving the appropriate loop invariant Q and evaluating Q at loop termination.

Function to return the value $x - y$ for $x, y \geq 0$.

Difference(nonnegative integer x; nonnegative integer y)

Local variables:
integers i, j

$i = 0$
$j = x$
while $i \neq y$ **do**
    $j = j - 1$
    $i = i + 1$
end while

// j now has the value $x - y$
return j
end function Difference

Attempt to prove the algorithm is correct using the **loop rule of inference**.

To do that, we must pick a Q, and prove it is a **loop invariant**. Then the loop rule of inference will allow us to state that Q ^ B' will be true after the loop. Q ^ B' should be chosen such that, *together*, they state "$j = x - y$".

Let Q equal "$j = x - i$"
Let B equal "$i \neq y$"

Attempt to prove Q is a loop invariant:

Let $Q(n)$ equal "$j_n = x - i_n$"

Prove $Q(0)$:

1.	$j_0 = x$	algorithm
2.	$i_0 = 0$	algorithm
3.	$j_0 = x - i_0$	hyp
4.	$j_0 = x - 0$	hyp
5.	$j_0 = x$	subtraction, 4
6.	(5) proves (3)	
7.	∴ $Q(0)$ is true	

Assume $Q(k)$: $j_k = x - i_k$

Prove $Q(k+1)$: $j_{k+1} = x - i_{k+1}$

1.	$j_{k+1} = j_k - 1$	algorithm
2.	$i_{k+1} = i_k + 1$	algorithm
3.	$i_k = i_{k+1} - 1$	subtract 1 from both sides of (2)
4.	$j_{k+1} = x - i_k - 1$	substitute $Q(k)$ into (1)
5.	$j_{k+1} = x - (i_{k+1} - 1) - 1$	substitute (3) into (4)
6.	$j_{k+1} = x - i_{k+1} + 1 - 1$	simplify (5)
7.	$j_{k+1} = x - i_{k+1}$	simplify (6)
8.	∴ $Q(k+1)$ is true	

We have proven that Q is a loop invariant.

Therefore, upon loop termination, the assertion $Q \wedge B'$ must be true, ie:

$$(j = x - i) \wedge (i = y)$$

Combining those statements yields:

$$j = x - y$$

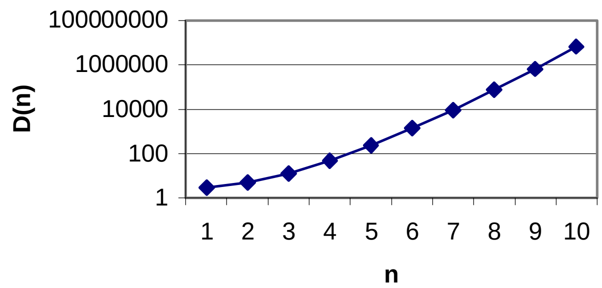
Therefore, the algorithm is provably correct.

§2.4

8. Write the first five values in the sequence:

$D(1) = 3$
$D(2) = 5$
$D(n) = (n-1)D(n-1) + (n-2)D(n-2)$ for $n > 2$

n	$D(n)$
1	3
2	5
3	13
4	49
5	235
6	1,371
7	9,401
8	74,033
9	658,071
10	6,514,903



15. Prove the given property of the Fibonacci numbers directly from the definition.

$F(n+6) = 4F(n+3) + F(n)$ for $n \geq 1$

Use 2nd principle of induction.

Prove for $P(1)$ and $P(2)$.

Prove for $P(1)$:

1. $F(n+6) = 4F(n+3) + F(n)$
2. $F(1+6) = 4F(1+3) + F(1)$
3. $F(7) = 4F(4) + F(1)$
4. $13 = 4(3) + 1$
5. $13 = 12 + 1$
6. $13 = 13$
7. *true*

Prove for $P(2)$:

1. $F(n + 6) = 4F(n + 3) + F(n)$
2. $F(2 + 6) = 4F(2 + 3) + F(2)$
3. $F(8) = 4F(5) + F(2)$
4. $21 = 4(5) + 1$
5. $21 = 20 + 1$
6. $21 = 21$
7. *true*

Assume that for all r, $1 \le r \le k$,

$F(r + 6) = 4F(r + 3) + F(r)$ inductive hypothesis

Now show for $P(k + 1)$, or

$F(k + 1 + 6) \stackrel{?}{=} 4F(k + 1 + 3) + F(k + 1)$
$F(k + 7) \stackrel{?}{=} 4F(k + 4) + F(k + 1)$

1.	$F(k + 7) = F(k + 5) + F(k + 6)$	Fibonacci
2.	*state* $F(k + 5)$ *in terms of inductive hypothesis*	
	a. Let $r = k – 1$	
	b. $F(r + 6) = 4F(r + 3) + F(r)$	ind. hyp.
	c. $F(k – 1 + 6) = 4F(k – 1 + 3) + F(k – 1)$	subst 2, 2a
	d. $F(k + 5) = 4F(k + 2) + F(k – 1)$	
3.	*state* $F(k + 6)$ *in terms of inductive hypothesis*	
	a. Let $r = k$	
	b. $F(r + 6) = 4F(r + 3) + F(r)$	ind. hyp.
	c. $F(k + 6) = 4F(k + 3) + F(k)$	subst 2, 2a
4.	$F(k + 7) = 4F(k + 2) + F(k – 1) + 4F(k + 3) + F(k)$	subst 2d, 3c, 1
5.	$F(k + 7) = 4F(k + 2) + 4F(k + 3) + F(k – 1) + F(k)$	comm, 4
6.	$F(k + 7) = 4[F(k + 2) + F(k + 3)] + F(k – 1) + F(k)$	dist, 5
7.	$F(k + 4) = F(k + 2) + F(k + 3)$	Fibonacci
8.	$F(k + 1) = F(k – 1) + F(k)$	Fibonacci
9.	$F(k + 7) = 4F(k + 4) + F(k + 1)$	subst 7, 8, 6
10.	$\therefore P(k+1)$ is true	

Therefore, by the 2nd inductive hypothesis, the property

$F(n + 6) = 4F(n + 3) + F(n)$

is true for all Fibonacci numbers where $n \ge 1$.

33. An amount of \$500 is invested in an account paying 10% interest compounded annually.

a. Write a recursive definition for $P(n)$, the amount in the account at the beginning of the nth year.

$P(1) = 500$
$P(n) = 1.1(P(n\text{-}1))$

b. After how many years will the account balance exceed \$700?

Five years:

year	principle
1	\$500.00
2	\$550.00
3	\$605.00
4	\$665.50
5	\$732.05

37. A collection W of strings of symbols is defined recursively by

1. a, b, and c belong to W.
2. If X belongs to W, so does $a(X)c$.

Which of the following belong to W?

a. $a(b)c$	**b.** $a(a(b)c)c$	**c.** $a(abc)c$	**d.** $a(a(a(a)c)c)c$	**e.** $a(aacc)c$
yes	yes	no	yes	no

---

§3.1

6a. Describe each of the following sets by giving a characterizing property:

$\{1, 2, 3, 4, 5\}$

$\{x \mid x$ is an integer and $0 < x < 6\}$

7b. Describe each of the following sets:

$\{ x \mid x \in \mathbb{N} \wedge (\exists y)(\exists z)(y \in \{0,1\}$ and $z \in \{3, 4\}$ and $y < x < z) \}$

Long version:

x is a non-negative integer ($x >= 0$), and
there is a *y* and a *z* where $y = 0$ or 1 and $z = 3$ or 4 and $y < x < z$.

$\therefore\ 0 < x < 4$
$\therefore\ x = \{ 1, 2, 3 \}$

Could not *x* be simply { 2 }? I don't think so, because the characterizing property specified the scope of *y* and *z* with $\exists$, not $\forall$. Therefore, the description is still true if *x* is 1 or 3.

10k. Let

$R = \{ 1, 3, \pi, 4.1, 9, 10 \}$ $S = \{ \{ 1 \}, 3, 9, 10 \}$
$T = \{ 1, 3, \pi \}$ $U = \{ \{ 1, 3, \pi \}, 1 \}$

Which of the following are true? For those that are not, why not?

$T \in U$ **true**

17. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

1.	$A \subseteq B$	hyp
2.	$B \subseteq C$	hyp
3.	$(\forall x)(x \in A \rightarrow x \in B)$	def. subset, 1
4.	$(\forall x)(x \in B \rightarrow x \in C)$	def. subset, 2
5.	t	ui, 3
6.	$t \in A$	temp hyp
	a. $t \in B$	mp, 5, 3
	b. $t \in C$	mp, 6, 4
7.	$(t \in A) \rightarrow (t \in C)$	temp hyp discharged
8.	$(\forall x)\,[(x \in A) \rightarrow (x \in C)]$	ug, 7
9.	$\therefore A \subseteq C$	def. subset, 8

35. Let

$A = \{ p, q, r, s \}$
$B = \{ r, t, v \}$
$C = \{ p, s, t, u \}$

be subsets of $S = \{ p, q, r, s, t, u, v, w \}$. Find

a. $B \cap C = \{ t \}$

b. $A \cup C = \{ p, q, r, s, t, u \}$

c. $C' = \{ q, r, v, w \}$

d. $A \cap B \cap C = \varnothing$

e. $B - C = \{ r, v \}$

f. $(A \cup C)' = \{ v, w \}$

g. $A \times B = \{$
$(p, r), (p, t), (p, v),$
$(q, r), (q, t), (q, v),$
$(r, r), (r, t), (r, v),$
$(s, r), (s, t), (s, v)$
$\}$

h. $(A \cup B) \cap C' = \{ q, r, v \}$

COSC 5060

System Concepts

Dr. Martin

4th Class Session

September 3, 2001

Mark Zieg

Read chapter 5 and complete the following problems to hand in:

p176 (§3.1): 5a, 11, 36k-L, 53a-c
p194 (§3.2): 3, 8, 19, 34, 52
p203 (§3.3): 3, 10, 13
p215 (§3.4): 8, 15, 33, 37
p225 (§3.5): 1g, 2

§3.1

5a. Describe the following set by listing its elements:

$\{ x \mid x \in \mathbb{N} \text{ and } x^2 - 5x + 6 = 0 \}$

1. $x^2 - 5x + 6 = 0$
2. $(x - 3)(x - 2) = 0$
3. $\{ 2, 3 \}$

11. Let

$A = \{ a, \{ a \}, \{ \{ a \} \} \}$ $B = \{ a \}$ $C = \{ \varnothing, \{ a, \{ a \} \} \}$

Which of the following are true? For those that are not, why not?

a. $B \subseteq A$
true

b. $B \in A$
true

c. $C \subseteq A$
false; although C_1 and C_2 are both subsets of A, $\{ C_1, C_2 \}$ is not.

d. $\varnothing \subseteq C$
true

e. $\varnothing \in C$
true

f. $\{ a, \{ a \} \} \in A$
false; A has no member matching this construction

g. $\{ a, \{ a \} \} \subseteq A$
true

h. $B \subseteq C$
false; C has elements which themselves contain B, but the subset operator does not default to recursive behaviour

i. $\{ \{ a \} \} \subseteq A$
true

36. Let

$A = \{ 2, 4, 5, 6, 8 \}$
$B = \{ 1, 4, 5, 9 \}$
$C = \{ x \mid x \in \mathbb{Z} \text{ and } 2 \le x < 5 \} = \{ 2, 3, 4 \}$

be subsets of $S = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$ [ie \d]. Find

k. $(B - A)' \cap (A - B)$

1. $(\{ 1, 4, 5, 9 \} - \{ 2, 4, 5, 6, 8 \})' \cap (\{ 2, 4, 5, 6, 8 \} - \{ 1, 4, 5, 9 \})$
2. $(\{ 1, 9 \})' \cap \{ 2, 6, 8 \}$
3. $\{ 0, 2, 3, 4, 5, 6, 7, 8 \} \cap \{ 2, 6, 8 \}$
4. $\{ 2, 6, 8 \}$
5. $A - B$

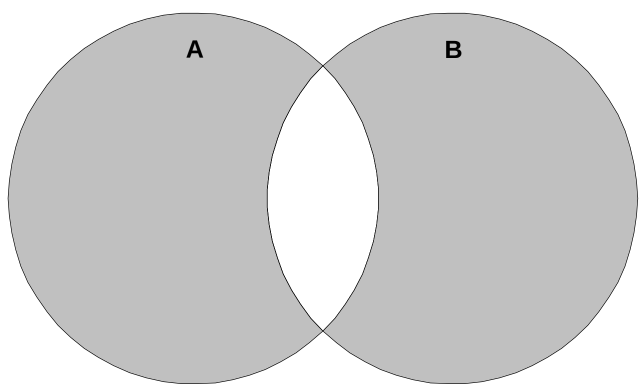
L. $(C' \cup B)'$

1. $(\{ 0, 1, 5, 6, 7, 8, 9 \} \cup \{ 1, 4, 5, 9 \})'$
2. $(\{ 0, 1, 4, 5, 6, 7, 8, 9 \})'$
3. $\{ 2, 3 \}$

53. A binary opration on sets called the *symmetric difference* is defined by:

$A \oplus B = (A - B) \cup (B - A)$

a. Draw a Venn diagram to illustrate $A \oplus B$.



b. For $A = \{ 3, 5, 7, 9 \}$ and $B = \{ 2, 3, 4, 5, 6 \}$, what is $A \oplus B$?

$\{ 2, 4, 6, 7, 9 \}$

c. Prove that $A \oplus B = (A \cup B) - (A \cap B)$ for arbitrary sets A and B.

Use proof method #2 from p173, *ie* pick arbitrary members from each side and prove they belong to the other side. That is, prove:

1. $x \in ((A - B) \cup (B - A)) \rightarrow x \in ((A \cup B) - (A \cap B))$
2. $x \in ((A \cup B) - (A \cap B)) \rightarrow x \in ((A - B) \cup (B - A))$

Proof #1:

1.	$x \in ((A - B) \cup (B - A))$	hyp
2.	$x \in (A - B)$ or $x \in (B - A)$	def'n union, 1
3.	$((x \in A)$ and $(x \notin B))$ or $((x \in B)$ and $(x \notin A))$	def'n subtr, 2
4.	For simplified notation, let:	
	a. P = $(x \in A)$	
	b. P' = $(x \notin A)$	
	c. Q = $(x \in B)$	
	d. Q' = $(x \notin B)$	
	e. From (3), we have (P ∧ Q') ∨ (Q ∧ P')	
	f. Try to prove (P ∨ Q) ∧ (P' ∨ Q')	
5.	(P ∧ Q') ∨ (Q ∧ P')	var subst, 3
6.	((P ∧ Q') ∨ Q) ^ ((P ∧ Q') ∨ P')	dist, 5
7.	(Q ∨ (P ∧ Q')) ^ (P' ∨ (P ∧ Q'))	comm, 5
8.	((Q ∨ P) ∧ (Q ∨ Q')) ∧ ((P' ∨ P) ∧ (P' ∨ Q'))	dist, 7
9.	((Q ∨ P) ∧ 1) ∧ (1 ∧ (P' ∨ Q'))	complement, 8
10.	(Q ∨ P) ∧ (P' ∨ Q')	identity, 9
11.	(P ∨ Q) ∧ (P' ∨ Q')	comm, 10
12.	$((x \in A)$ or $(x \in B))$ and $((x \notin A)$ or $(x \notin B))$	var subst, 11
13.	$((x \in A)$ or $(x \in B))$ and $((x \in A)$ and $(x \in B))'$	DeMorgan, 12
14.	$x \in (A \cup B)$ and $((x \in A)$ and $(x \in B))'$	def'n ∪, 13
15.	$x \in (A \cup B)$ and $(x \in (A \cap B))'$	def'n ∩, 14
16.	$x \in (A \cup B)$ and $x \notin (A \cap B)$	dbl neg, 15
17.	$x \in ((A \cup B) - (A \cap B))$	def'n subtr, 16

Proof #2 reads the exact same way, only in reverse (steps 17 back up to 1).

---

§3.2

3. A video game on a microcomputer is begun by making selections from each of three menus. The first menu (number of players) has four selections, the second (level of play) has eight, and the third menu (speed) has six. In how many configurations can the game be played?

$4 \times 8 \times 6 = 192$

8. *A*, *B*, *C*, and *D* are nodes on a computer network. There are two paths between *A* and *C*, two between *B* and *D*, three between *A* and *B*, and four between *C* and *D*. Along how many routes can a message from *A* to *D* be sent?

Given...	*Assuming...*	*Then...*
AC = 2	AD = 0	ACD = 2 × 4 = 8
BD = 2	BC = 0	ABD = 3 × 2 = 6
AB = 3	CB = 0	Total = 14
CD = 4	DA = 0	
	DB = 0	
	DC = 0	

19. A new car can be ordered with a choice of 10 exterior colors; 7 interior colors; automatic, 3-speed, or 5-speed transmission; with or without air conditioning; with or without power steering; and with or without the option package that contains the power door lock and rear window defroster. How many different cars can be ordered?

E = 10	I = 7	T = 3
A = 2	P = 2	O = 2

Combinations = E × I × T × A × P × O = 1680

34. Regarding the set of binary strings of length 8 (ie, /^[01]{8}$/), how many have 1 as the second digit?

Note that "second" isn't defined as leading from the left or right, but it doesn't matter – the answer would be "half of the total", obviously, which would be:

$$2^8 \div 2 = 2^7 = 128$$

52. Regarding a hand of cards, where one hand consists of a single card drawn from a standard 52-card deck, with flowers on the back, and a second card drawn from a standard 52-card deck, with birds on the back, how many hands consist of a pair of aces?

There are four aces in each deck (SHCD), each of which may be paired with any of four aces in the other deck, so there are 4 × 4 = 16 possible hands consisting of a pair of aces.

That's *assuming* that the ornament on the back of the card is considered sufficient to distinguish { Ace-Spades-Flower, Ace-Hearts-Bird } from { Ace-Spades-Bird, Ace-Hearts-Flower }, as indicated by the note given to problem 51...

§3.3

3. Quality control in a factory pulls 40 parts with paint, packaging, or electronics defects from an assembly line. Of these, 28 had a paint defect, 17 had a packaging defect, 13 had an electronics defect, 6 had both paint and packaging defects, 7 and both packaging and electronic defects, and 10 had both paint and electronics defects. Did any part have all three types of defect?

Let: A = { parts with paint defects }
K = { parts with packaging defects }
E = { parts with electronics defects }

$$|A \cup K \cup E| = 40$$
$$|A| = 28$$
$$|K| = 17$$
$$|E| = 13$$
$$|A \cap K| = 6$$
$$|K \cap E| = 7$$
$$|A \cap E| = 10$$

From the three-set version of the Principle of Inclusion and Exclusion (p199), we know that:

1. $|A \cup K \cup E| = |A| + |K| + |E| - |A \cap K| - |K \cap E| - |A \cap E| + |A \cap K \cap E|$
2. $40 = 28 + 17 + 13 - 6 - 7 - 10 + |A \cap K \cap E|$
3. $40 - 28 - 17 - 13 + 6 + 7 + 10 = |A \cap K \cap E|$
4. $5 = |A \cap K \cap E|$

Answer: **yes**

10. You are developing a new bath soap, and you hire a public opinion survey group to do some market research for you. The group claims that, in its survey of 450 consumers, the following were named as important factors in purchasing bath soap:

Odor	425	$\|O\|$
Lathering ease	397	$\|L\|$
Natural ingredients	340	$\|N\|$
Odor and lathering ease	284	$\|O \cap L\|$
Odor and natural ingredients	315	$\|O \cap N\|$
Lathering ease and natural ingredients	219	$\|L \cap N\|$
All three factors	147	$\|O \cap L \cap N\|$

Should you have confidence in these results? Why or why not?

From the three-set version of the Principle of Inclusion and Exclusion (p199), we know that:

1. $|O \cup L \cup N| = |O| + |L| + |N| - |O \cap L| - |O \cap N| - |L \cap N| + |O \cap L \cap N|$
2. 450 = 425 + 397 + 340 – 284 – 315 – 219 + 147
3. 450 = 491
4. contradiction!

Therefore, no, the marketing results should not be trusted, because their math doesn't hold up. ±3% margin of error is considered reasonable for many polls and surveys, yet these marketing figures exceed ±9%.

13. How many cards must be drawn from a standard 52-card deck to guarantee 2 cards of the same suit?

    5, using the Pigeonhole Principle (and common sense).

---

§3.4

8. **a.** Stock designations on an exchange are limited to three letters. How many different designations are there?

Note: this question hinges on the interpretation of "limited", ie "are limited to having <u>precisely</u> three letters," or "are limited to having <u>no more than</u> three letters."

Assuming interpretation #1:

$26^3 = 17{,}576$

Assuming interpretation #2:

$26^3 + 26^2 + 26 = 18{,}278$

b. How many different designations are there if letters cannot be repeated?

Assuming interpretation #1:

$$\begin{aligned} P(26, 3) &= 26! \div (26-3)! \\ &= 26! \div 23! \\ &= 26 * 25 * 24 \\ &= 15{,}600 \end{aligned}$$

Assuming interpretation #2:

P(26, 3) + P(26, 2) + P(26, 1)
15,600 + 26 * 25 + 26
15,600 + 26^2
15,600 + 676
16,276

15. Compute $C(n, n-1)$. Explain why $C(n, n-1) = C(n, 1)$.

$$
\begin{aligned}
C(n, n-1) &= \frac{n!}{(n-1)!(n-(n-1))!} \\
&= \frac{n!}{(n-1)!(n-n+1)!} \\
&= \frac{n!}{(n-1)!(1)!} \\
&= \frac{n!}{(n-1)!} \\
&= n
\end{aligned}
$$

$C(n, n-1)$ equals $C(n, 1)$ because both counting problems have the same cardinality.

$C(n, n-1)$ is counting the number of ways you can take all elements of a set *but one*; in other words, you iterate through instances of the original set of *n* elements, taking a different one *out* each time. Obviously, the number of different elements which you can take out of the set is equal to the number of elements in the set.

This is numerically equivalent to $C(n, 1)$, which simply computes the number of ways you can take a single element out of the original set, which again is directly equal to the cardinality of the set.

33. Concerning a 5-card hand from a standard 52-card deck:

How many hands contain a straight flush (five consecutive cards, for example, ace, 2, 3, 4, 5 of the same suit)?

> *Note: I'm not a frequent poker player, so I'm assuming, from the way this question was stated, that a "straight flush" <u>does</u> include A-2-3-4-5, but does <u>not</u> include a "royal flush", ie 10-J-Q-K-A.*

Straight flushes: $9 \times 4 = 36$

A♠-2♠-3♠-4♠-5♠	A♥-2♥-3♥-4♥-5♥	A♣-2♣-3♣-4♣-5♣	A♦-2♦-3♦-4♦-5♦
2♠-3♠-4♠-5♠-6♠	2♥-3♥-4♥-5♥-6♥	2♣-3♣-4♣-5♣-6♣	2♦-3♦-4♦-5♦-6♦
3♠-4♠-5♠-6♠-7♠	3♥-4♥-5♥-6♥-7♥	3♣-4♣-5♣-6♣-7♣	3♦-4♦-5♦-6♦-7♦
4♠-5♠-6♠-7♠-8♠	4♥-5♥-6♥-7♥-8♥	4♣-5♣-6♣-7♣-8♣	4♦-5♦-6♦-7♦-8♦
5♠-6♠-7♠-8♠-9♠	5♥-6♥-7♥-8♥-9♥	5♣-6♣-7♣-8♣-9♣	5♦-6♦-7♦-8♦-9♦
6♠-7♠-8♠-9♠-10♠	6♥-7♥-8♥-9♥-10♥	6♣-7♣-8♣-9♣-10♣	6♦-7♦-8♦-9♦-10♦
7♠-8♠-9♠-10♠-J♠	7♥-8♥-9♥-10♥-J♥	7♣-8♣-9♣-10♣-J♣	7♦-8♦-9♦-10♦-J♦
8♠-9♠-10♠-J♠-Q♠	8♥-9♥-10♥-J♥-Q♥	8♣-9♣-10♣-J♣-Q♣	8♦-9♦-10♦-J♦-Q♦
9♠-10♠-J♠-Q♠-K♠	9♥-10♥-J♥-Q♥-K♥	9♣-10♣-J♣-Q♣-K♣	9♦-10♦-J♦-Q♦-K♦

37. Fourteen copies of a code module are to be executed in parallel on identical processors organized into two communicating clusters, *A* and *B*. Cluster *A* contains 16 processors and cluster *B* contains 32 processors.

Find the number of ways to choose the processors if cluster *A* has three failed processors and cluster *B* has two failed processors.

> Well, the problem doesn't state that the module must be load-balanced (evenly or otherwise) between the two clusters – in fact, problem 35 suggests rather strongly that it is a legitimate outcome to have all modules running on a single cluster.
>
> Therefore, out of the original clusters *A* and *B*, it would seem that we end up with an unordered collection of 43 working processors, 14 of which must be used for this deployment.
>
> Therefore, there are $C(43, 14)$ unique [working] deployment configurations, ie:

1. $C(43, 14)$
2. $\dfrac{43!}{14!(43-14)!}$
3. $\dfrac{43!}{14!29!}$
4. $\dfrac{43\times 42\times\cdots\times 30\times(29\times 28\times\cdots\times 1)}{14!29!}$
5. $\dfrac{43\times42\times41\times40\times39\times38\times37\times36\times35\times34\times33\times32\times31\times30}{14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}$
6. $43\times41\times3\times38\times37\times5\times34\times2\times31$
7. 78,378,960,360

Ie, there are about 80 billion possibilities.

---

§3.5

1g. Expand the expression using the binomial theorem:

$$(2p-3q)^4$$
$$1(2p)^4(-3q)^0+4(2p)^3(-3q)^1+6(2p)^2(-3q)^2+4(2p)^1(-3q)^3+1(2p)^0(-3q)^4$$
$$16p^4-96p^3q+216p^2q^2-216pq^3+81q^4$$

2. Find the fourth term in the expansion of $(a + b)^{10}$.

Let $n = 10$
$k = 4$

$$C(n,k-1)a^{n-(k-1)}b^{k-1}$$
$$C(10,3)a^7b^3$$
$$\frac{10!a^7b^3}{3!(10-3)!}$$
$$\frac{10!a^7b^3}{3!7!}$$
$$\frac{10\times9\times8\times a^7b^3}{3!}$$
$$\frac{10\times9\times8\times a^7b^3}{3\times2\times1}$$
$$10\times3\times4a^7b^3$$
$$120a^7b^3$$

COSC 5060

Dr. Martin

Mark Zieg

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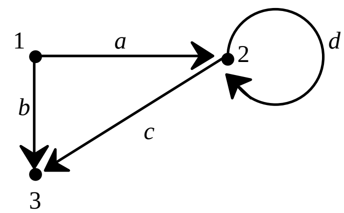
5th Class Session

September 10th, 2001

Read chapter 6 and complete the following problems:
5.1: *1, 11, 33, 40, 56, 68, 71b*
5.2: *10, 14a, 19, 28*
5.3: *3, 8, 20*
5.4: *2, 8*

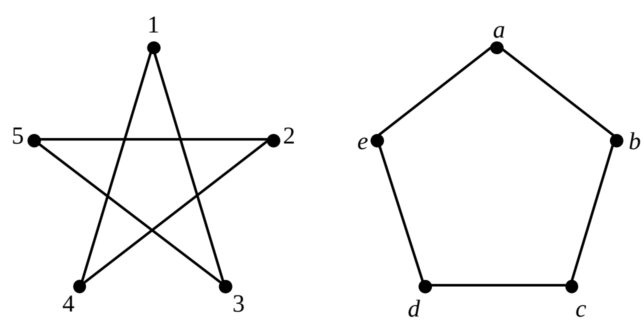
Section 5.1

1. Give the function *g* that is part of the formal definition of the directed graph shown.



$g(a) =$ *a*: 1-2
b: 1-3
c: 2-3
d: 2-2

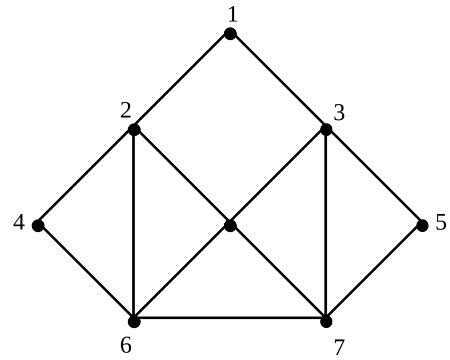
11. Decide if the two graphs are isomorphic. If so, give the function or functions that establish the isomorphism; if not, explain why.



$f_1 =$ 1 → *a*
2 → *d*
3 → *b*
4 → *e*
5 → *c*

$f_2 =$ 1, 4 → *a*, *e*
1, 3 → *a*, *b*
2, 5 → *d*, *c*
2, 4 → *d*, *e*
3, 5 → *b*, *c*

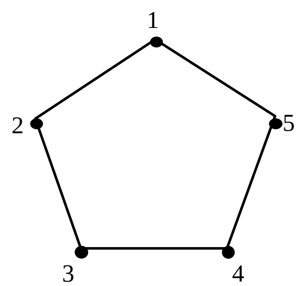
33. **Write the adjacency matrix for the graph:**



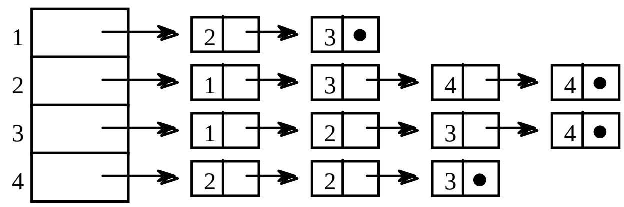
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

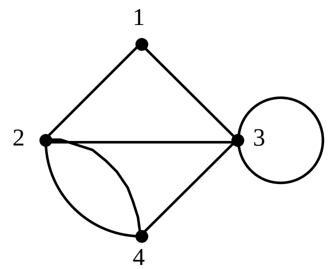
40. **Draw the graph represented by the adjacency matrix:**

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

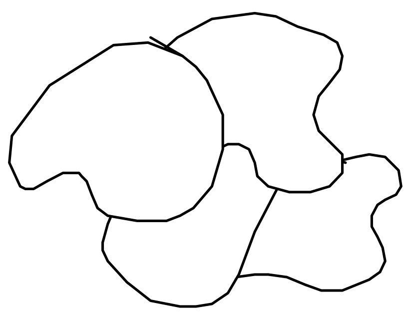


Draw the undirected graph represented by the adjacency list in the accompanying figure:

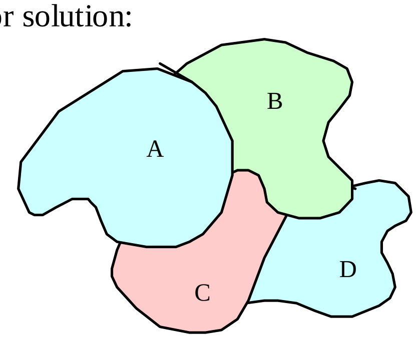




68. **Show that a coloring of the accompanying map requires three colors and no more than three colors.**



It's simple to prove that the map requires *no more than* three colors, by demonstrating a 3-color solution:



In order to show that the map requires at least three colors, we can label the four regions *A* through *D*, as shown above, and demonstrate that all possible one- and two-color solutions result in adjacent regions of the same color.

One color:

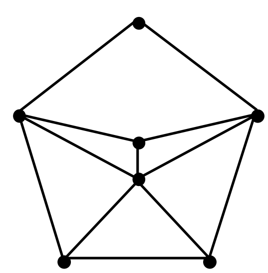
A and B are adjacent

Two colors:

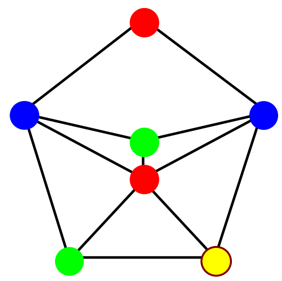
Given two colors and four regions, there are 16 possible colorations to choose from. Each of them results in at least one case of adjoining same-color regions (indicated by a slash through the affected cells). Let the colors be represented by the integer values '0' and '1':

A	*B*	*C*	*D*
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

71b. A *coloring* of a graph is an assignment of a color to each node of the graph in such a way that no two adjacent nodes have the same color. The *chromatic number* of a graph is the smallest number of colors needed to achieve a coloring. Find the chromatic number of the following graph:



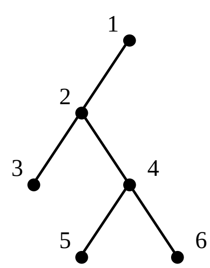
The smallest possible number is 4:



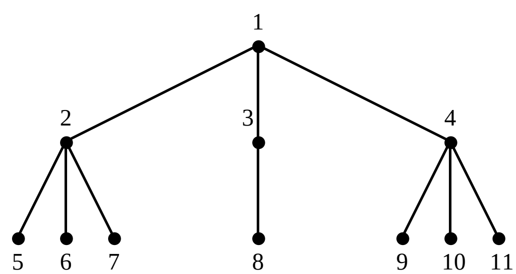
5.2

10. Draw the binary tree represented by the left child-right child representation of the figure. (1 is the root.)

	Left child	Right child
1	2	0
2	3	4
3	0	0
4	5	6
5	0	0
6	0	0

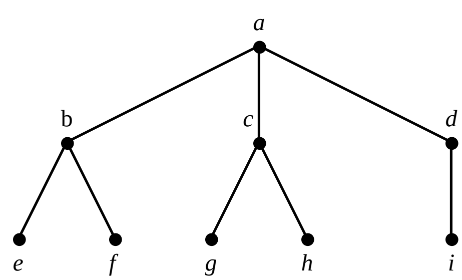


14b. For the tree in the figure, write the leftmost child-right sibling array representation described in Exercise 13.



	Leftmost child	Right sibling
1	2	0
2	5	3
3	8	4
4	9	0
5	0	6
6	0	7
7	0	0
8	0	0
9	0	10
10	0	11
11	0	0

19. **Write the list of nodes resulting from a *preorder* traversal, an *inorder* traversal, and a *postorder* traversal of the tree.**



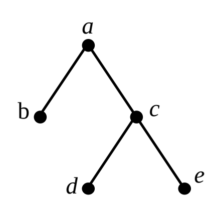
Preorder: *a, b, e, f, c, g, h, d, i*
Inorder: *e, b, f, a, g, c, h, i, d*
Postorder: *e, f, b, g, h, c, i, d, a*

28. **Draw a tree whose preorder traversal is**

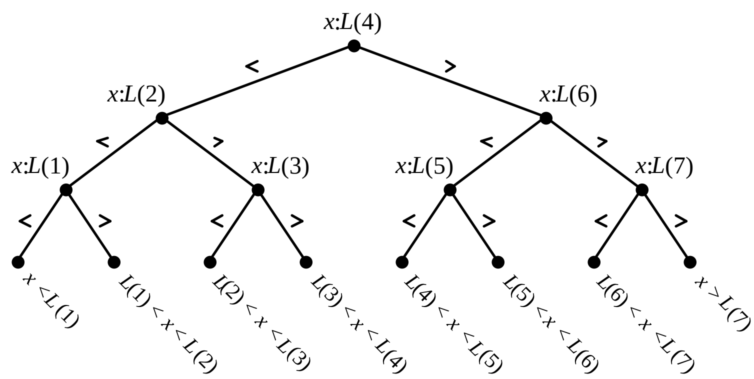
a, b, c, d, e

and whose inorder traversal is

b, a, d, c, e



3. **Draw the decision tree for binary search on a list of seven elements. What is the depth of the tree?**

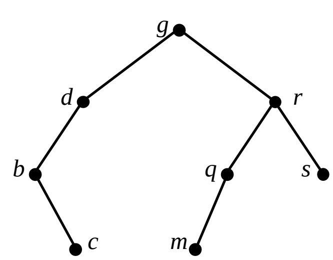


Tree depth = 3

8a. **Given the data**

g, d, r, s, b, q, c, m

Construct the binary search tree. What is the depth of the tree?



Depth = 3

8b. **Find the average number of comparisons done to search for an item that is known to be in the list using binary tree search on the tree of part (a). (*Hint:* Find the number of comparisons for each of the items.)**

2.75

20. **One of eight coins is counterfeit and is either too heavy or too light. The problem is to identify the counterfeit coin and determine whether it is heavy or light.**

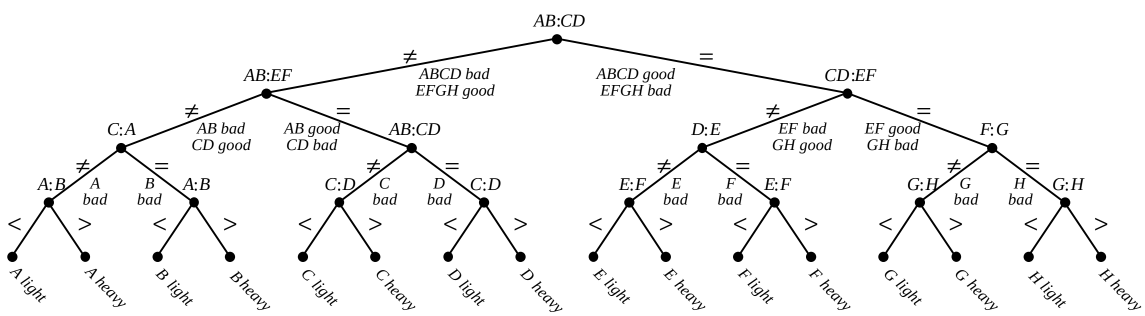
a. **What is the number of final outcomes (the number of leaves in the decision tree)?**

16 (ie, eight coins × (heavy or light))

b. **Find a lower bound on the number of comparisons required to solve this problem in the worst case.**

4

c. **Devise an algorithm that meets this lower bound (draw its decision tree).**



5.4

2. **Given the codes**

Character	***b***	***h***	***q***	***w***	***%***
Encoding scheme	1000	1001	0	11	101

Decode the sequences

a. 10001001101101

b h % %

b. 11110

w w q

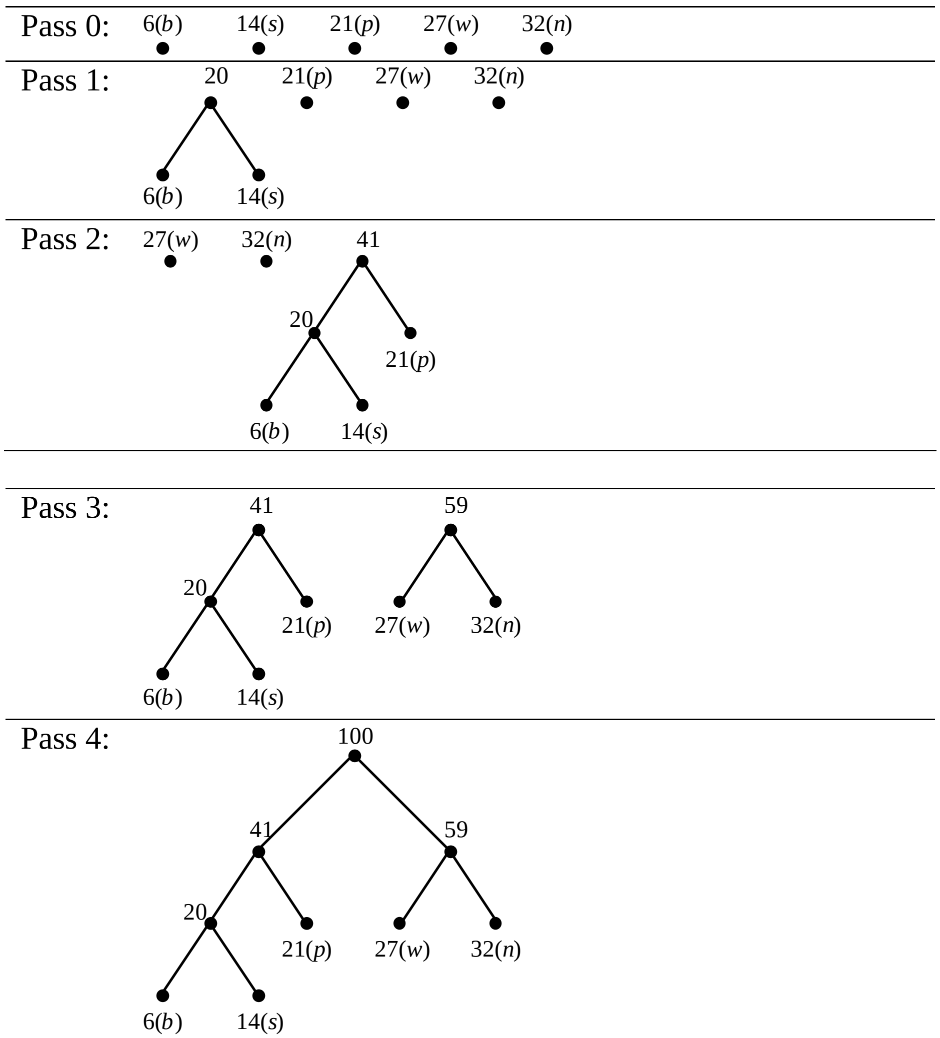
c. 01001111000

q h w b

8.

a. Construct the Huffman tree for the following characters and frequencies:

Character	***B***	***n***	***p***	***s***	***w***
Frequency	6	32	21	14	27



b. Find the Huffman codes for these characters

Character	*B*	*n*	*p*	*s*	*w*
Encoding scheme	000	11	01	001	10

COSC 5060

System Concepts

Dr. Martin

Mid-term

September 17, 2001

Mark Zieg

Complete the following problems to hand in:

p15 (§1.1): 11e
p30 (§1.2): 9, 25, 35
p44 (§1.3): 15a, 16d, 17d
p56 (§1.4): 10, 14
p93 (§2.1): 21, 40, 46, 47
p106 (§2.2): 10, 37
p118 (§2.3): 4, 5
p137 (§2.4): 5, 29, 52, 78
p177 (§3.1): 43, 47, 59c, 59e, 59f
p195 (§3.2): 22
p203 (§3.3): 6, 7
p215 (§3.4): 2, 58, 68
p225 (§3.5): 1g

§1.1

11e. Construct truth tables for the following wffs. Note any tautologies or contradictions.

$(A \to B) \to [(A \vee C) \to (B \vee C)]$

A	B	C	$A \to B$	$A \vee C$	$B \vee C$	$(A \vee C) \to (B \vee C)$	$(A \to B) \to [(A \vee C) \to (B \vee C)]$
F	F	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	T	T	T	T	T	T	T

The final column is all "true", so the implication is a tautology.

§1.2

9. Use propositional logic to prove that the argument is valid.

$(A \to B) \wedge [B \to (C \to D)] \wedge [A \to (B \to C)] \to (A \to D)$

1.	$A \to B$	hyp
2.	$B \to (C \to D)$	hyp
3.	$A \to (B \to C)$	hyp
4.	A	temp hyp
a.	B	mp, 1, 4
b.	$C \to D$	mp, 2, 4a
c.	$B \to C$	mp, 3, 4
d.	C	mp, 4a, 4c
e.	D	mp, 4b, 4d
5.	$A \to D$	temp hyp discharged

25. Use propositional logic to prove the arguments valid; you may use any of the rules in Table 1.14 or any previously proven exercise.

$(A \vee B) \wedge (A \to C) \wedge (B \to C) \to C$

1.	$A \vee B$	hyp
2.	$A \to C$	hyp
3.	$B \to C$	hyp
4.	$A' \to B$	imp, 1
5.	$A' \to C$	hs, 4, 3
6.	$C' \to A$	cont, 2
7.	$C' \to C$	hs, 6, 2
8.	$(C')' \vee C$	imp, 7
9.	$C \vee C$	dn, 8
10.	C	self, 9

35. Using propositional logic, including the rules in Table 1.14, prove that each argument is valid. Use the statement letters shown.

If Jose took the jewelry or Mrs. Krasov lied, then a crime was committed. Mr. Krasov was not in town. If a crime was committed, then Mr. Krasov was in town. Therefore Jose did not take the jewelry. (J, L, C, T)

$((J \vee L) \to C) \wedge T' \wedge (C \to T) \to J'$

1.	$J \vee L \to C$	hyp
2.	T'	hyp
3.	$C \to T$	hyp
4.	$C' \vee T$	imp, 3
5.	$(J \vee L)' \vee C$	imp, 1
6.	$(J' \wedge L') \vee C$	DeMorgan, 5
7.	$T' \wedge (C' \vee T)$	con, 2, 4
8.	$(T' \wedge C') \vee (T' \wedge T)$	dist, 7
9.	$(T' \wedge C') \vee 0$	comp, 8
10.	$T' \wedge C'$	ident, 9
11.	C'	sim, 10
12.	$C' \wedge ((J' \wedge L') \vee C)$	con, 11, 6
13.	$(C' \wedge (J' \wedge L')) \vee (C' \wedge C)$	dist, 12
14.	$(C' \wedge (J' \wedge L')) \vee 0$	comp, 13
15.	$C' \wedge (J' \wedge L')$	ident, 14
16.	$J' \wedge L'$	sim, 15
17.	J'	sim, 16

§1.3

15a. Explain why the wff is valid:

$$(\forall x)(\forall y)A(x, y) \leftrightarrow (\forall y)(\forall x)A(x, y)$$

The wff is valid because universal existence – the truth of the existence of the entire set of possible values for *x* and *y* – is not predicated upon order or sequence. In the left-hand side of the wff, the relation *A* is said to be true for **all** *x* and **all** *y*, not just those values of *x* or *y* which meet a certain criteria – including criteria of order within the statement. Therefore, the order of the existential qualifiers on the right-hand side of the wff may be rearranged arbitrarily, since they are not issuing constraints or limitations upon successive operators in left-to-right evaluation. This is really stating a sort of communitive property of universal existence: that if something is said to always exist, under all circumstances and conditions, then it always exists whether or not another proposition is said to be true, and without dependency on other extants.

16d. Give interpretations to prove that each of the following wffs is not valid:

$$(\forall x)[\ A(x)\ ]' \leftrightarrow [\ (\forall x)A(x)\ ]'$$

Simplify the expression:

1. $(\forall x)[A(x)]' \leftrightarrow [(\forall x)A(x)]'$
2. $(\forall x)[A(x)]' \leftrightarrow (\exists x)[A(x)]'$
3. $(\ (\forall x)[A(x)]' \rightarrow (\exists x)[A(x)]'\ ) \wedge\ (\ (\exists x)[A(x)]' \rightarrow (\forall x)[A(x)]'\ )$

Now we only have to disprove $(\exists x)[A(x)]' \rightarrow (\forall x)[A(x)]'$ to disprove the whole statement.

Disproof by example:

the domain is all integers
$A(x)$: $x^2 > x$
$A(x)'$: $x^2 <= x$
$(\exists x)[A(x)]'$ is easily shown where *x* is zero or one ($0^2 \leq 0$, and $1^2 \leq 1$).
However, $(\forall x)[A(x)]'$ is just as easily disproven where *x* is 2 or –2 ($2^2 \nleqslant 2$, and $(-2)^2 \nleqslant (-2)$).

Therefore, the wff is invalid, since we have shown an interpretation where $(\exists x)[A(x)]'$ is true, yet $(\forall x)[A(x)]'$ is false.

17d. *Decide* whether the following wff is valid or invalid. *Justify* your answer.

$$(\forall x)[\, P(x) \vee Q(x)\,] \rightarrow (\forall x)P(x) \vee (\exists y)Q(y)$$

I believe that the wff is valid. I believe it to be probably valid because the consequent seems to follow logically from the antecedent. The antecedent states that, for every possible *x*, either *P* or *Q* must be true for that *x*.

This leaves only five possible cases:

1. *P* is true for all *x*, while *Q* is false for all *x* ($P \wedge Q'$)
2. *Q* is true for all *x*, while *P* is false for all *x* ($P' \wedge Q$)
3. *P* and *Q* are both true for all *x* ($P \wedge Q$)
4. *P* is true for some *x*, and *Q* is true for all other *x* ($P \oplus Q$)
5. *P* is true for some *x*, *Q* is true for some other *x*, and *P* and *Q* are both true for some *x* ($P \vee Q$)

The antecedent is basically postulating that any of the five cases may be true. On surface examination, it would appear that the consequent holds true for any of the five cases:

1. $(\forall x)P(x)$ is true in this case
2. $(\exists y)Q(y)$ is true in this case
3. $(\forall x)P(x)$ and $(\exists y)Q(y)$ are both true in this case
4. $(\exists y)Q(y)$ is true in this case
5. $(\exists y)Q(y)$ is true in this case

Therefore, the wff appears to be valid, although my deductions have not been formulated into a conclusive proof.

10. Either prove that the wff is a valid argument or give an interpretation in which it is false.

$$(\exists x)\ [R(x) \vee S(x)] \rightarrow (\exists x)R(x) \vee (\exists x)S(x)$$

Start by re-writing the conclusion as:

$$[(\exists x)R(x)]' \rightarrow (\exists x)S(x)$$

9. $(\exists x)\ [R(x) \vee S(x)]$	hyp
10. $[(\exists x)R(x)]'$	temp hyp
a. $(\forall x)R(x)'$	neg, 2
b. $R(a)'$	ui, 2a
c. $R(a) \vee S(a)$	ei, 1
d. $R(a)' \rightarrow S(a)$	imp, 2c
e. $S(a)$	mp, 2b, 2d
f. $(\exists x)S(x)$	eg, 2e
11. $[(\exists x)R(x)]' \rightarrow (\exists x)S(x)$	temp hyp discharged

14. Either prove that the wff is a valid argument or give an interpretation in which it is false.

$$(\forall x)P(x) \vee (\exists x)Q(x) \rightarrow (\forall x)[\ P(x) \vee Q(x)\ ]$$

The wff would seem to be invalid, using the following interpretation:

The domain is all days of the year
$P(x)$ = x is my birthday (Nov 20)
$Q(x)$ = the WTC was bombed on that day (Sep 11)

This is true for the antecedent, since, although my birthday does *not* fall on every day of the year, nonetheless, and regretably, there does exist a day on which the WTC was bombed (Sep 11).

However, the consequent is false, since it requires that *every* day of the year be either my birthday or a terrorist bombing. Biology and cosmology make the first impossible, and history (and CNN) showed that the latter tragedy did not, in fact, reoccur indefinitely (only, as it turned out, twice).

§2.1

21. Prove the given statement:

> If the product of two integers is *not* divisible by an integer n, then neither integer is divisible by n.

Or, stated algebraeically,

$$(xy|n)' \rightarrow (x|n)' \wedge (y|n)'$$

Proof by contraposition:

$$((x|n)' \wedge (y|n)')' \rightarrow ((xy|n)')'$$

1.	$((x\|n)' \wedge (y\|n)')'$	hyp
2.	$(x\|n) \vee (y\|n)$	DeMorgan, 1
3.	pick factor a such that $a = x \div n$	
4.	$(x = a \times n) \vee (y\|n)$	subst, 2, 3
5.	pick factor b such that $b = y \div n$	
6.	$(x = a \times n) \vee (y = b \times n)$	subst, 4, 5
7.	$(xy = a \times n \times y) \vee (y = b \times n)$	mult both sides of left by y
8.	$(xy = a \times n \times y) \vee (xy = x \times b \times n)$	mult both sides of right by x
9.	$(xy\|n) \vee (xy\|n)$	def'n of a factor
10.	$(xy\|n)$	self, 9
11.	$((xy\|n)')'$	dn, 10

40. Prove or disprove the given statement:

> For every prime number n, $n + 4$ is prime.

Disproof by contradiction:

> 2 is prime.
> $2 + 4 = 6$
> 6 is not prime (divisible by 2 and 3).

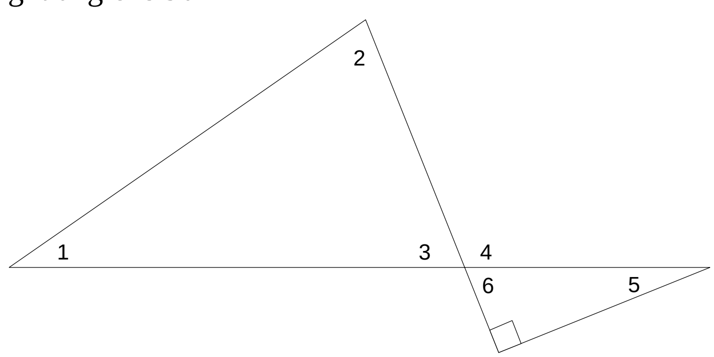
46. Prove or disprove the given statement:

The sum of a rational number and an irrational number is irrational.

Proof by contradiction:

1. Exercise #44 proves that the sum of any two rational numbers is guaranteed to be rational (easily shown using basic addition of fractions).
2. It is furthermore easily shown that, if the sum of any two rational numbers is rational, so then is the *difference* of any two rational numbers (since any rational number p/q can be multiplied by the rational number –1/1, and then the additive proof can be confirmed).
3. Pick arbitrary rational and irrational numbers *R* and *I*.
4. Let their sum be *S*, ie $S = I + R$
5. Assume that *S* is rational
6. From this assumption, $S - R$ should equal *I*, our irrational number.
7. However, from step #2, the difference $S - R$ should be a rational number, not irrational. Steps #3 and #6 represent a contradiction: *I* is defined as being irrational, yet is found to be equal to a difference which has been proven to always be rational.
8. Therefore, the hypothesis in step 5 must be false.

47. For the following exercise, use the accompanying figure and the following facts from geometry:
1. The interior angles of a triangle sum to 180º
2. Vertical angles (opposite angles formed when two lines intersect) are the same size
3. A straight angle is 180º
4. A right angle is 90º



Prove that the measure of $\angle 4$ is the sum of $\angle 1$ and $\angle 2$.

1. $\angle 3 + \angle 4 = 180^\circ$	rule #3
2. $\angle 1 + \angle 2 + \angle 3 = 180^\circ$	rule #1
3. $\angle 4 = 180^\circ - \angle 3$	subtr, 1
4. $\angle 1 + \angle 2 = 180^\circ - \angle 3$	subtr, 2
5. $\angle 1 + \angle 2 = \angle 4$	subst, 3, 4

§2.2

10. Use mathematical induction to prove that the statement is true for every positive integer *n*.

$$1^4 + 2^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

To prove this via the 1st principle of mathematical induction, we must complete the following sub-proofs:

3. prove P(1)
4. assuming P(k), prove P(k+1)

Direct proof of #1:

$$1^4 = \frac{1(1+1)(2+1)(3+3-1)}{30}$$

$$1 = \frac{2(3)(5)}{30}$$

$$1 = 1$$

true

Direct proof of #2:

$$1^4 + 2^4 + \cdots + k^4 + (k+1)^4 = \frac{(k+1)(k+1+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30}$$

$$[1^4 + 2^4 + \cdots + k^4] + (k+1)^4 = \frac{(k+1)(k+2)(2k+3)(3(k^2+2k+1)+3k+2)}{30}$$

$$\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + [k^4+4k^3+6k^2+4k+1] = \frac{(k^2+3k+2)(2k+3)(3k^2+9k+5)}{30}$$

$$\frac{(2k^3+3k^2+k)(3k^2+3k-1)}{30} + k^4+4k^3+6k^2+4k+1 = \frac{(2k^3+6k^2+4k+3k^2+9k+6)(3k^2+9k+5)}{30}$$

$$(2k^3+3k^2+k)(3k^2+3k-1)+30k^4+120k^3+180k^2+120k+30 = (2k^3+6k^2+4k+3k^2+9k+6)(3k^2+9k+5)$$

$$(6k^5+6k^4-2k^3+9k^4+9k^3-3k^2+3k^3+3k^2-k)+30k^4+120k^3+180k^2+120k+30 = (2k^3+9k^2+13k+6)(3k^2+9k+5)$$

$$6k^5+45k^4+130k^3+180k^2+119k+30 = 6k^5+18k^4+10k^3+27k^4+81k^3+45k^2+39k^3+117k^2+65k+18k^2+54k+30$$

$$6k^5+45k^4+130k^3+180k^2+119k+30 = 6k^5+45k^4+130k^3+180k^2+119k+30$$

$$1 = 1$$

true

37. Prove that the statement is true for every positive integer.

$3^{2n} + 7$ is divisible by 8.

To prove this via the 1st principle of mathematical induction, we must complete the following sub-proofs:

5. prove P(1)
6. assuming P(k), prove P(k+1)

Direct proof of #1:

$$3^{2(0)} + 7 = 8x$$
$$3^{0} + 7 = 8x$$
$$1 + 7 = 8x$$
$$8 = 8x$$
$$1 = x$$
true

Direct proof of #2:

$$3^{2(k+1)} + 7 = 8y$$
$$3^{2k+2} + 7 = 8y$$
$$(3^{2k})(3^{2}) + 7 = 8y$$
$$(8x - 7)(9) + 7 = 8y$$
$$72x - 63 + 7 = 8y$$
$$72x - 56 = 8y$$
$$8(9x - 8) = 8y$$
true

§2.3

4. Prove that the pseudocode program segment is correct by proving the loop invariant *Q* and evaluating *Q* at loop termination.

Function to compute and write out quotient *q* and remainder *r* when *x* is divided by *y*, $x \geq 0$, $y \geq 1$

```
Divide( nonnegative integer x; positive integer y )
   Local variables:
       nonnegative integers q, r
   q = 0
   r = x
   while r ≥ y do
       q++
       r -= y
   end while
   // q and r are now the quotient and remainder
   write( "The quotient is %d and the remainder is %d", q, r )
end function Divide
```

Q: $x = q \times y + r$

Attempt to prove the algorithm is correct using the **loop rule of inference**.

To do that, we must pick a *Q*, and prove it is a **loop invariant**. Then the loop rule of inference will allow us to state that Q ^ B' will be true after the loop. Q ^ B' should be chosen such that, *together*, they state "$j = x - y$".

Let *Q* equal "$x = q \times y + r$"
Let *B* equal "$r \geq y$"

Attempt to prove Q is a loop invariant:

Let $Q(n)$ equal "$x = q_n \times y + r_n$"

Prove $Q(0)$:

8. $x = q_0 \times y + r_0$
9. $x = 0 \times y + x$
10. $x = x$
11. $\therefore$ $Q(0)$ is true

Assume $Q(k)$: $x = q_k \times y + r_k$

Prove $Q(k+1)$: $x = q_{k+1} \times y + r_{k+1}$

9. $x = q_k \times y + r_k$	hyp
10. $q_{k+1} = q_k + 1$	algorithm
11. $r_{k+1} = r_k - y$	algorithm
12. $q_k = q_{k+1} - 1$	subtraction, 2
13. $r_k = r_{k+1} + y$	addition, 3
14. $x = (q_{k+1} - 1) \times y + r_k$	substitution, 1, 4
15. $x = (q_{k+1} - 1) \times y + r_{k+1} + y$	substitution, 6, 5
16. $x = q_{k+1} \times y - y + r_{k+1} + y$	dist. mult, 7
17. $x = q_{k+1} \times y + r_{k+1}$	target
18. $\therefore Q(k+1)$ is true	

We have proven that Q is a loop invariant.

Therefore, upon loop termination, the assertion $Q \wedge B'$ must be true, ie:

$$(x = q \times y + r) \wedge (r < y)$$

Combining those statements yields:

$$x = q \times y + r$$

Therefore, the algorithm is provably correct.

5. Use the Euclidean algorithm to find the greatest common divisor of the given numbers:

(2420, 70)

$$70 \overline{)2420} \;\; 34; \quad 2420 - 2380 = 40$$

$$40 \overline{)70} \;\; 1; \quad 70 - 40 = 30$$

$$30 \overline{)40} \;\; 1; \quad 40 - 30 = 10$$

$$10 \overline{)30} \;\; 3; \quad 30 - 30 = 0$$

$GCD = 10$

§2.4

5. Write the first five values in the sequence:

$T(1) = 1$
$T(n) = n \times T(n-1) \quad \text{for } n \geq 2$

n	T(n)
1	**1**
2	**2**
3	**6**
4	**24**
5	**120**
6	720
7	5,040
8	40,320
9	362,880
10	3,628,800

29. Prove the given property of the Fibonacci numbers directly from the definition.

$F(n+6) = 4F(n+3) + F(n) \quad \text{for } n \geq 1$

1. F(n+6) = F(n+4) + F(n+5)	def
2. F(n+5) = F(n+3) + F(n+4)	def
3. F(n+4) = F(n+2) + F(n+3)	def
4. F(n+3) = F(n+1) + F(n+2)	def
5. F(n+2) = F(n) + F(n+1)	def
6. F(n+6) = F(n+2) + F(n+3) + F(n+3)+F(n+4)	subst 2, 3 into 1
7. F(n+6) = 2F(n+3) + F(n+2) + F(n+4)	simplify, 6
8. F(n+6) = 2F(n+3) + F(n+2) + F(n+2) + F(n+3)	subst 3 into 7
9. F(n+6) = 3F(n+3) + F(n+2) + F(n+2)	simplify, 8
10. F(n+6) = 3F(n+3) + F(n+2) + F(n) + F(n+1)	subst 5 into 9
11. F(n+6) = 3F(n+3) + F(n+3) + F(n)	subst 4 into 10
12. ∴ F(n+6) = 4F(n+3) + F(n)	simplify

52. Write the body of the recursive function to compute $S(n)$ for the given sequence S.

1, 3, 9, 27, 51, …

```
int S( int n ) {
   if( n < 1 ) {
      fprintf( stderr, “S(n) not defined for n < 1” );
      exit();
   } else if( n == 1 ) {
      return 1;
   } else {
      return 3 * S(n-1);
   }
}
```

78. In an account that pays 8% annually, $1000 is deposited. At the end of each year, an additional $100 is deposited into the account. What is the account worth at the end of 7 years (that is, at the beginning of the 8th year)?

Year	Balance
1	1,000.00
2	1,180.00
3	1,374.40
4	1,584.35
5	1,811.10
6	2,055.99
7	2,320.47
8	2,606.10

---

§3.1

43. An internet search engine has the following set of URL references in its database:

A = automobiles for sale
with subsets:
B = used cars
C = Fords
D = Buicks
E = pre-1995 models

You want to search for all references to used Fords or Buicks that are 1995 or later models.

Write the set expression that represents your query.

Obviously, this query isn't sufficiently precise to be converted into set notation. There are at least three likely interpretations of the query, each resulting in different equations:

"used (Fords or Buicks) that are 1995 or later"

$B \cap (C \cup D) \cap E$

"((used Fords) or Buicks) that are 1995 or later"

$((B \cap C) \cup D) \cap E$

"(used Fords) or (Buicks that are 1995 or later)"

$(B \cap C) \cup (D \cap E)$

47. Prove that

$A \subseteq (A \cup B)$

where A and B are arbitrary sets.

1.	A	hyp
2.	B	hyp
3.	t	ui
4.	$(t \in A) \vee (t \in B) \leftrightarrow t \in (A \cup B)$	def'n union
5.	$t \in A$	temp hyp
	a. $t \in (A \cup B)$	mp, 4, 5
6.	$t \in A \rightarrow t \in (A \cup B)$	temp hyp discharged
7.	$(\forall x)(x \in A \rightarrow x \in (A \cup B))$	ug, 6
8.	$A \subseteq (A \cup B)$	def'n subset

59c. *A*, *B*, and *C* are subsets of a set *S*. Prove the following set identities by using previously proved identities, including those in Exercises 56-58 [and on p171].

$(A - B) - C = (A - C) - B$

1.	$(A - B) - C$	starting point
2.	$(A \cap B') - C$	def'n set diff
3.	$(A \cap B') \cap C'$	def'n set diff
4.	$A \cap (B' \cap C')$	assoc (p171 2b)
5.	$A \cap (C' \cap B')$	comm (p171 1b)
6.	$(A \cap C') \cap B'$	assoc (p171 2b)
7.	$(A \cap C') - B$	def'n set diff
8.	$(A - C) - B$	def'n set diff

59e. *A*, *B*, and *C* are subsets of a set *S*. Prove the following set identities by using previously proved identities, including those in Exercises 56-58 [and on p171].

$(A - B) - C = (A - C) - (B - C)$

Well, I tried proving this until my hair turned grey, and couldn't figure it out. Finally I turned the quivalence around, and found that much easier to show. Since equivalence is bidirectional, the following proof is sufficient:

1. (A – C) – (B – C) starting point
2. (A ∩ C') – (B – C) def set sub, 1
3. (A ∩ C') – (B ∩ C') def set sub, 2
4. (A ∩ C') ∩ (B ∩ C')' def set sub, 3
5. (A ∩ C') ∩ ((B ∩ C')')' dn, 4
6. (A ∩ C') ∩ (B' ∪ C) DeMorgan (56b), 5
7. ((A ∩ C') ∩ B') ∪ ((A ∩ C') ∩ C) dist, 6
8. (A ∩ (C' ∩ B')) ∪ (A ∩ (C' ∩ C)) assoc, 7
9. (A ∩ (C' ∩ B')) ∪ (A ∩ ∅) comp, 8
10. (A ∩ (C' ∩ B')) ∪ ∅ 58c, 9
11. (A ∩ (C' ∩ B')) 4a, 10
12. (A ∩ (B' ∩ C')) comm, 11
13. ((A ∩ B') ∩ C') assoc, 12
14. (A ∩ B') – C def set sub, 13
15. (A – B) – C def set sub, 14

59f. *A*, *B*, and *C* are subsets of a set *S*. Prove the following set identities by using previously proved identities, including those in Exercises 56-58 [and on p171].

A – (A – B) = A ∩ B

1. A – (A – B) starting point
2. A – (A ∩ B') def'n set diff
3. A ∩ (A ∩ B')' def'n set diff
4. A ∩ (A' ∪ B) DeMorgan
5. (A ∩ A') ∪ (A ∩ B) dist
6. ∅ ∪ (A ∩ B) comp (5b)
7. A ∩ B ident (4a)

---

§3.2

22. A customer at a fast-food restaurant can order a hamburger with or without mustard, ketchup, pickle, or onion; a fish sandwich with or without lettuce, tomato, or tartar sauce; and a choice of three kinds of soft drinks or two kinds of milkshakes. How many different orders are possible if a customer can order at most one hamburger, one fish sandwich, and one beverage, but can order less?

Scratch:

{0,1} hamburger (M, K, P, O) = 2^4 opts = 16 opts = 17 choices (incl none)
{0,1} fish (L, T, S) = 2^3 opts = 8 opts = 9 choices (incl none)
{0,1} drink (S1 or S2 or S3 or M1 or M2) = 6 choices (incl none)

Total: 17 * 9 * 6 = 918

---

§3.3

6. From the 83 students who want to enroll in CS 320, 32 have completed CS 120, 27 have completed CS 180, and 35 have completed CS 215. Of these, 7 have completed both CS 120 and CS 180, 16 have completed CS 180 and CS 215, and 3 have completed CS 120 and CS 215. Two students have completed all three courses. The prerequisite for CS 320 is completion of one of CS 120, CS 180, or CS 215. How many students are not eligible to enroll?

   A = "completed CS 120"
   B = "completed CS 180"
   C = "completed CS 215"
   W = "want to take CS 320"

   $|W| = 83$

   $|A| = 32$
   $|B| = 27$
   $|C| = 35$
   $|A \cap B| = 7$
   $|B \cap C| = 16$
   $|A \cap C| = 3$
   $|A \cap B \cap C| = 2$

   Total number of students who have taken A, B, or C (and are thus eligible for CS 320) can be determined using the 3-set version of the Principle of Inclusion and Exclusion:

   $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
   $|A \cup B \cup C| = 32 + 27 + 35 - 7 - 16 - 3 + 2$
   $|A \cup B \cup C| = 59 + 35 - 26 + 2 = 95 - 25 = 70$

   Total NOT able to enroll = $|W| - |A \cup B \cup C|$
   $83 - 70$
   13 losers

7. Among a bank's 214 customers with checking or savings accounts, 189 have checking accounts, 73 have regular savings accounts, 114 have money market savings accounts, and 69 have both checking and regular savings accounts. No customer is allowed to have both regular savings accounts and money market savings accounts.

a. How many customers have both checking and money market savings accounts?

93

b. How many customers have a checking account but no savings account?

16

Scratch:

P = people (customers)
A = checking accounts
B = regular savings accounts
C = money market savings accounts

$\lvert P \rvert$	=	214
$\lvert A \rvert$	=	189
$\lvert B \rvert$	=	73
$\lvert C \rvert$	=	114
$\lvert A \cap B \rvert$	=	69
$\lvert B \cap C \rvert$	=	0 (by rule)
$\lvert A \cap C \rvert$	=	93 (see proof, below)
$\lvert A \cap B \cap C \rvert$	=	0 (by implication; $B \cap C$ is already $\varnothing$)/

a. Find $\lvert A \cap C \rvert$

$\lvert A \cup B \cup C \rvert = \lvert A \rvert + \lvert B \rvert + \lvert C \rvert - \lvert A \cap B \rvert - \lvert B \cap C \rvert - \lvert A \cap C \rvert + \lvert A \cap B \cap C \rvert$
$214 = 189 + 73 + 114 - 69 - 0 - \lvert A \cap C \rvert + 0$
$\lvert A \cap C \rvert = 189 + 73 + 114 - 69 - 214$
$\lvert A \cap C \rvert = 93$

b. Find $\lvert A - (B \cup C) \rvert$
$A - (B' \cap C')'$
$A - (B' - C)'$
$189 - (141 - 100)'$
$189 - (41)'$
$189 - 173$
16

---

§3.4

2. How many batting orders are possible for a nine-man baseball team?

$9! = 362{,}880$

58. In a 5-card hand from a standard 52-card deck, how many ways are there to have exactly 3 jacks and 2 hearts?

Ways to have 3 jacks:

4 (four suits, one of which is missing from each possibility, equals $C(4,3)$)

Ways to have 2 hearts:

C(13, 2) including the jack, or C(12, 2) if the jack is already in play

Combining these possibilities with the specifications "5-card hand" and "*exactly* three jacks", we see that we can never count the Jack of Hearts as one of the two hearts, because then we would either have four jacks (J♠-J♦-J♣-J♥-*x*♥) or only four cards (J_{S1}-J_{S2}-J♥-*x*♥).

Likewise, it would seem that we can never use the Jack of Hearts as a Jack. If we did, then either we would have three hearts (ie, J♠-J♦- J♥-*x*♥-*y*♥), or only four cards. This would mean that the number of ways to arrive at exactly 3 jacks would be exactly ONE: J♠-J♦-J♣.

Note that I have deliberately discounted the case in which the Jack of Hearts *is* counted as both a Jack and a Heart, by filling out the hand with a fifth card which is neither a Jack nor a Heart, ie J♠-J♦-J♥-Q♥-7♣. Under one interpretation of the question, that *is* a "5-card hand from a standard 52-card deck…[with] **'exactly'** 3 jacks and 2 hearts."

It is ironic that, in this context, it is the term "*exactly*" which is itself ambiguous: does this mean that "there should be no more or less than 3 jacks, and no more or less than 2 hearts, but no other constraints are imposed [other than the 5-card requirement]"; or does it mean, "the hand should be composed exclusively of 3 jacks and 2 hearts, with no cards other than jacks or hearts in the hand at all?" I have chosen to use the second interpretation; other interpretations may yield different results.

Therefore, the total should be (to my thinking):

$$\text{JackCombinations} \times \text{HeartCombinations}$$

$1 \times C(12,2)$

$\left(\frac{12!}{2!(12-2)!}\right)$

$\left(\frac{12!}{2 \times 10!}\right)$

$\left(\frac{12 \times 11}{2}\right)$

6×11

66

68. At a birthday party, a mother serves a cookie to each of 8 children. There are plenty of chocolate chip, peanut butter, and oatmeal cookies.

a. In how many ways can each child get one cookie? (We don't care which child gets which kind.)

n = 3
r = 8
C(8+3-1, 8)
45

b. In how many ways can each child get one cookie if at least one of each kind of cookie is given out?

n = 3
r = 5
C(5+3-1, 5)
21

c. In how many ways can each each child get one cookie if no one likes oatmeal cookies?

n = 2
r = 8
C(8+2-1, 8)
9

d. In how many ways can each child get one cookie if two children insist on getting peanut butter?

n = 3
r = 6
C(6+3-1, 6)
28

e. In how many ways can each child get one cookie if there are only two chocolate chip cookies?

$n = 2$		$n = 3$
$r = 6$		$r = 2$
C(6+2-1, 6)	+	C(2+3-1, 2)
7	+	6
13		

§3.5

1g. Expand the expression using the binomial theorem:

$$(2p-3q)^4$$

$$1(2p)^4(-3q)^0+4(2p)^3(-3q)^1+6(2p)^2(-3q)^2+4(2p)^1(-3q)^3+1(2p)^0(-3q)^4$$

$$16p^4-96p^3q+216p^2q^2-216pq^3+81q^4$$

COSC 5060

Dr. Martin

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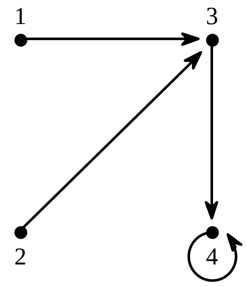
6th Class Session

September 24th, 2001

Read chapter 7 and complete the following problems:
6.1: 2, 4, 6, 22
6.2: 4, 11, 16, 25
6.3: 2, 18
6.4: 8, 12, 18, 23, 26
6.5: 2, 6, 8

Section 6.1

2. Find the adjacency matrix and adjacency relation for the graph in the figure.

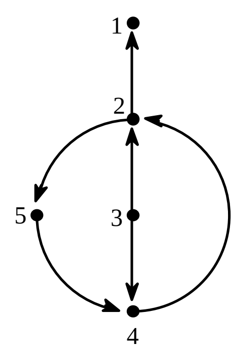


$\rho = \{ (1, 3), (2, 3), (3, 4), (4, 4) \}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

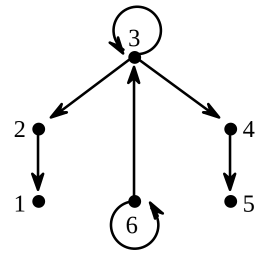
4. Find the corresponding directed graph and adjacency relation for the following adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$\rho = \{ (2, 1), (2, 5), (3, 2), (3, 4), (4, 2), (5, 4) \}$

6. Given the adjacency relation ρ = { (2, 1), (3, 2), (3, 3), (3, 4), (4, 5), (6, 3), (6, 6) } on the set N = { 1, 2, 3, 4, 5, 6 }, find the corresponding directed graph and adjacency matrix.



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

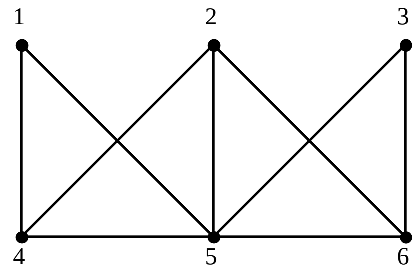
22. Compute the reachability matrix **R** for exercise 6 by using the formula:

$$\mathbf{R} = \mathbf{A} \vee \mathbf{A}^{(2)} \vee \cdots \vee \mathbf{A}^{(n)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Section 6.2

4. Determine whether the graph in the specified figure has an Euler path by using the theorem on Euler paths.



No, the graph cannot have an Euler path, because there are four odd nodes (2, 4, 5, and 6). By the theorem, the graph can only have an Euler path if the count of odd nodes is either zero or two.

11. Draw the adjacency matrix for the graph of Exercise 4. In applying algorithm *EulerPath*, what is the value of *total* after the fourth pass through the **while** loop?

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

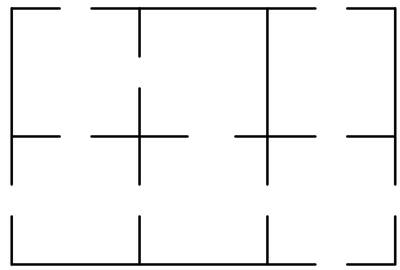
After the fourth pass through the loop, *total* equals “2”.

16. Decide by *trial and error* whether a Hamiltonian circuit exists for the graph of exercise 4.

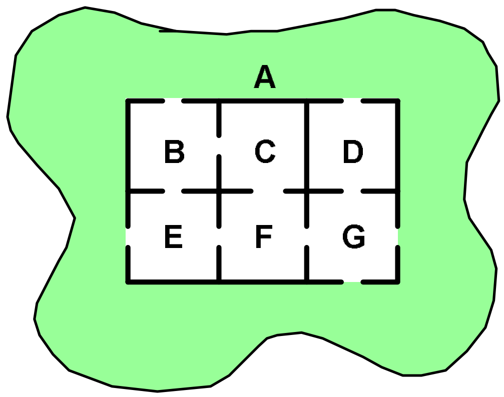
Yes, there are numerous Hamiltonian circuits to traverse the graph, including:

1-4-2-5-3-6
1-5-3-6-2-4
etc

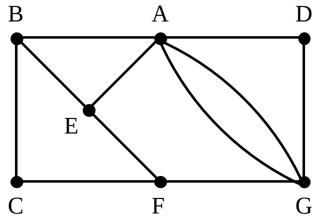
25. Is it possible to walk in and out of each room in the house, shown in the accompany figure, so that each door of the house is used exactly once? Why or why not?



If we label the rooms of the house, as well as the surrounding "lawn" (or "sidewalk" for you urbanites), then we have the following figure:



This figure can be represented with the following graph, with nodes representing rooms (or the lawn), and arcs indicating doors:



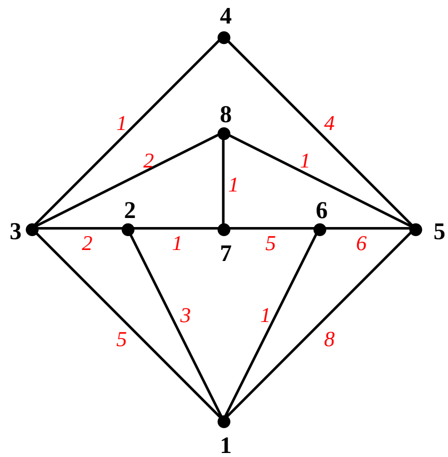
By Euler's theorem, we could traverse this graph (walk through the house), tracing every arc (passing through every door) exactly once, *if and only if* the number of odd nodes (rooms) is zero or two.

However, looking at this graph, we see that there are four "odd" nodes (A, B, E, and F).

Therefore, the house cannot be walked in the manner described.

Section 6.3

2. Using the accompanying graph, apply algorithm *ShortestPath* (Dijkstra's algorithm) for a path from *node 3 to node 6*. Show the values for p and IN and the d-values and s-values for each pass through the **while** loop. Write out the nodes in the shortest path and its distance.



Pass 0:

$IN = \{3\}$

	1	2	3	4	5	6	7	8
d	5	2	∞	1	∞	∞	∞	2
s	3	3	-	3	-	-	-	3

Pass 1:

$p = 4$
$IN = \{3, \mathbf{4}\}$

	1	2	3	4	5	6	7	8
d	5	2	**2**	1	**5**	∞	∞	2
s	3	3	**4**	3	**4**	-	-	3

Pass 2:

$p = 2$
$IN = \{\mathbf{2}, 3, 4\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	5	∞	**3**	2
s	3	3	4	3	4	-	**2**	3

Pass 3:

$p = 8$
$IN = \{2, 3, 4, \mathbf{8}\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	**3**	∞	3	2
s	3	3	4	3	**8**	-	2	3

Pass 4:

$p = 7$
$IN = \{2, 3, 4, \mathbf{7}, 8\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	3	**8**	3	2
s	3	3	4	3	8	**7**	2	3

Pass 5:

$p = 5$
$IN = \{2, 3, 4, \mathbf{5}, 7, 8\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	3	8	3	2
s	3	3	4	3	8	7	2	3

Pass 6:

$p = 1$
$IN = \{\mathbf{1}, 2, 3, 4, 5, 7, 8\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	3	**6**	3	2
s	3	3	4	3	8	**1**	2	3

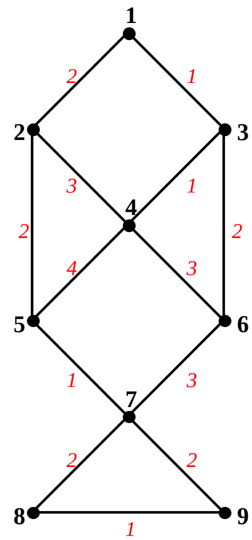
Pass 7:

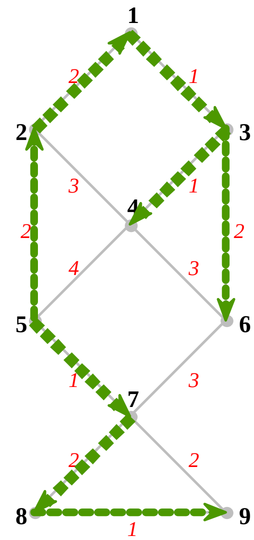
$p = 6$
$IN = \{1, 2, 3, 4, 5, \mathbf{6}, 7, 8\}$

	1	2	3	4	5	6	7	8
d	5	2	2	1	3	6	3	2
s	3	3	4	3	8	1	2	3

Shortest path:	3 – 1 – 6
distance:	2 hops
cumulative weight:	6

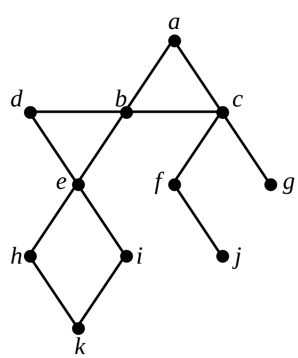
18. Find a minimal spanning tree for the graph in the specified figure.





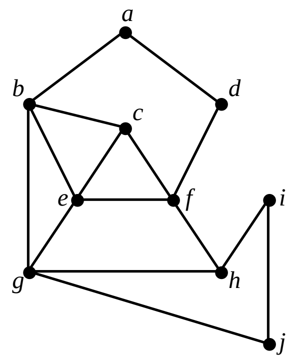
Section 6.4

8. Write the nodes in a depth-first search of the graph in the accompanying figure, beginning with the node *e*.



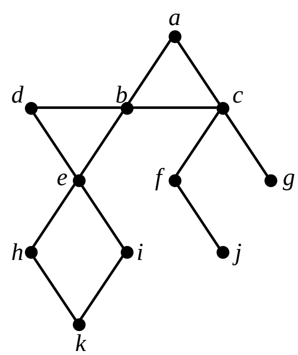
e-i-k-h-b-d-c-a-f-j-g

12. Write the nodes in a breadth-first search of the graph in the accompanying figure, beginning with the node *c*.



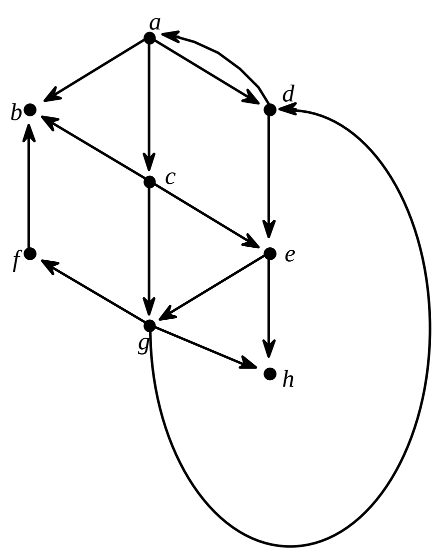
c-f-b-e-a-d-h-g-j-i

18. Write the nodes in a breadth-first search of the graph in the accompanying figure, beginning with the node *e*.



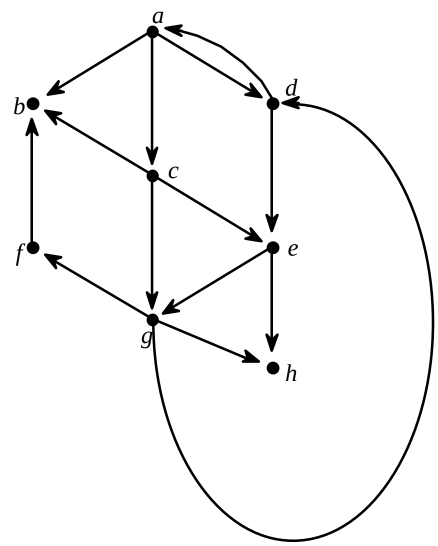
e-b-d-i-h-c-a-k-f-g-j

23. Write the nodes in a depth-first search of the graph in the accompanying figure, beginning with the node *f*.



f, b

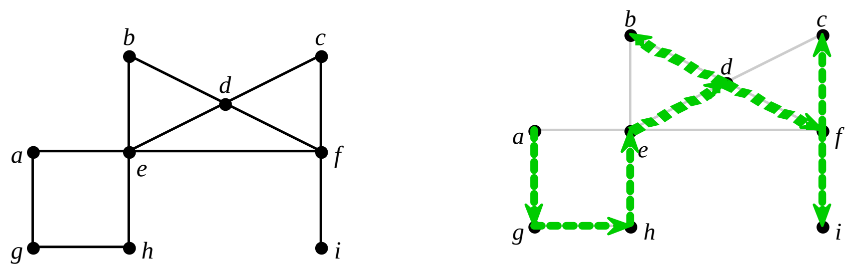
26. Write the nodes in a breadth-first search of the graph in the accompanying figure, beginning with the node *f*.



f, *b*

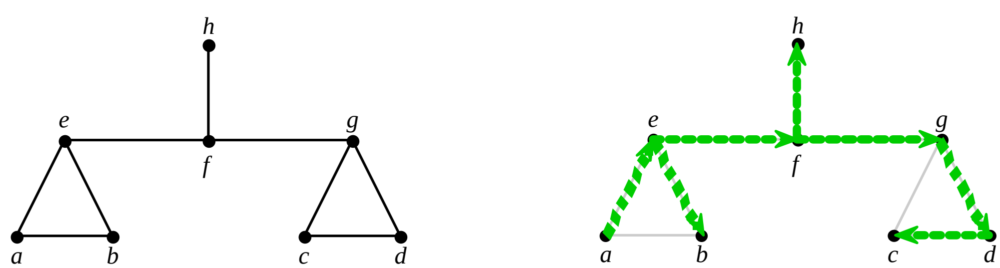
Section 6.5

2. Draw the depth-first search trees, where node *a* is the starting node of the depth-first search. Identify the back arcs.



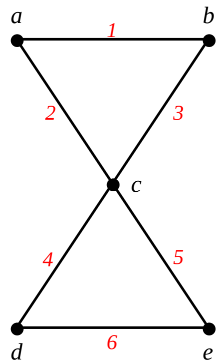
back arcs: *e*-*a*, *b*-*e*, *d*-*c*

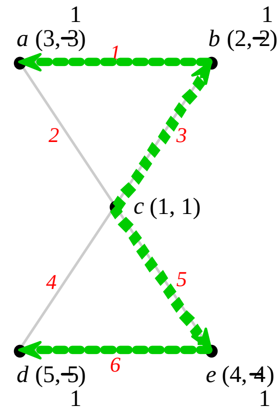
6 Draw the depth-first search trees, where node *a* is the starting node of the depth-first search. Identify the back arcs.



back arcs: *a*-*b*, *g*-*c*

8. Use algorithm *ArtPoint* to find the articulation points. Label *TreeNumber* and *BackNumber* for each node, both as first assigned and as changed. Draw the biconnected components of the graph.





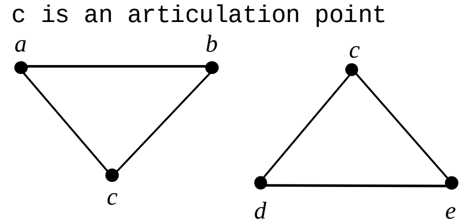
visited: *c*, *b*, *a*, *e*, *d*

tree edges: 3, 1, 5, 6

stdout:

```
c is an articulation point
c is an articulation point
```

Biconnected components:



COSC 5060

Dr. Martin

Mark Zieg

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7th Class Session

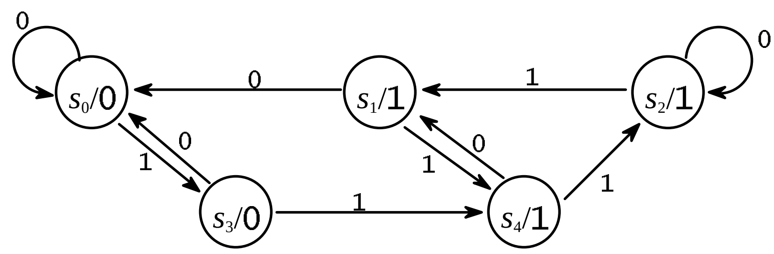
Oct 1st, 2001

Complete the following problems to hand in:
8.2: 2a, 2c, 4, 8, 23a, 24b, 25e, 35
8.3: 2, 4, 13
8.4: 7, 13, 20a

Section 8.2

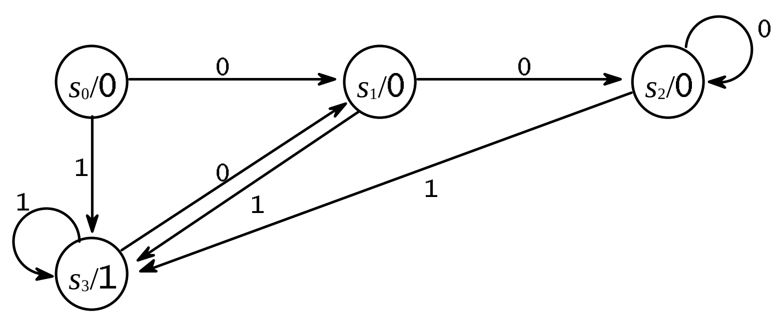
2.

a. For the machine described in Exercise 1(a), find all input sequences yielding an output sequence of 0011110.



input: 11(101∨010)0(0∨1)
POSIX: /^11(101|010)0[01]$/

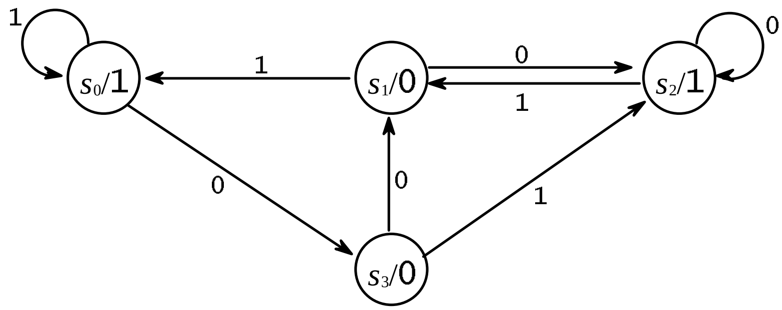
c. For the machine described in Exercise 1(c), what will be the output for an input sequence $a_1\ a_2\ a_3\ a_4\ a_5$ where $a_i \in \{0,1\}$, $1 \le i \le 5$?



output: $0a_1\ a_2\ a_3\ a_4\ a_5$ where $a_i \in \{0,1\}$, $1 \le i \le 5$
POSIX: /^0[01]{5}$/

4. Write the state table for the machine, and compute the output sequence for the given input sequence.

1101100



State table:

	Next State		
	Present Input		
Present State	0	1	Output
s_0	s_3	s_0	1
s_1	s_2	s_0	0
s_2	s_2	s_1	1
s_3	s_1	s_2	0

Output:

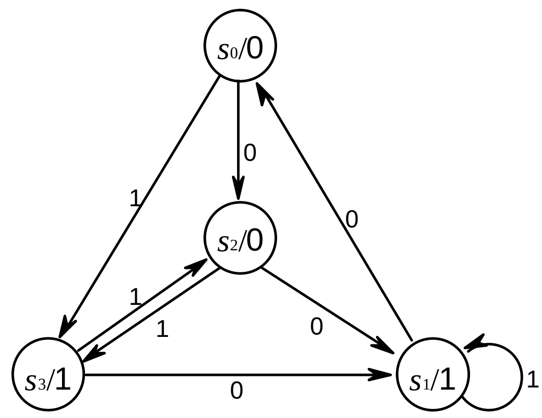
11101011

8. Draw the state graph for the machine, and compute the output sequence for the given input sequence.

0011

	Next State		
	Present Input		
Present State	0	1	Output
s_0	s_2	s_3	0
s_1	s_0	s_1	1
s_2	s_1	s_3	0
s_3	s_1	s_2	1

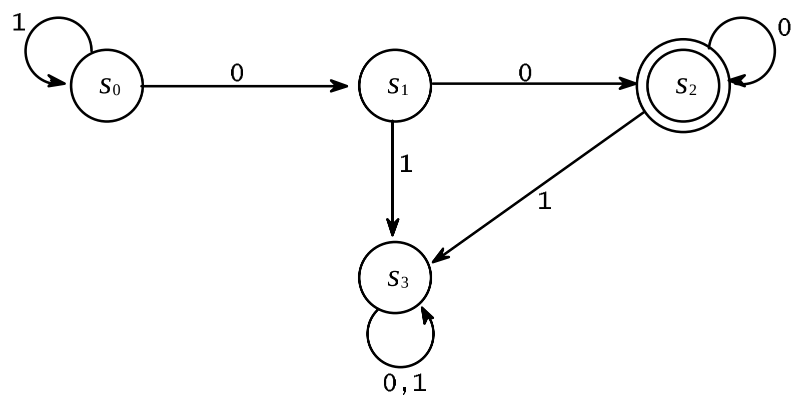
State diagram:



Output:

00111

23a. Give a regular expression for the set recognized by the finite-state machine in the accompanying figure.



Recognized input strings:

1*000*
POSIX: /^1*00+$/

24b. Give a regular expression for the set recognized by each finite-state machine in the accompanying table.

Present State	Next State Present Input 0	1	Output
s_0	s_3	s_1	1
s_1	s_1	s_2	1
s_2	s_2	s_2	0
s_3	s_0	s_2	0

Recognized input strings:

(00)* ∨ (00)*10*

POSIX: /^(00)*(10*)?$/

25e. Give a regular expression for the following set:

set of all strings of *a*'s and *b*'s where each *a* is followed by two *b*'s

Recognized input strings:

(*b**(*abb*)**b**)*

POSIX: /^(*b**(*abb*)**b**)*$/

35. Minimize *M*.

Present State	Next State Present Input 0	1	Output
0	5	3	1
1	5	2	0
2	1	3	0
3	2	4	1
4	2	0	1
5	1	4	0

0-equivalent states: {0, 3, 4}, {1, 2, 5}

1-equivalent states: {0, 3, 4}, {1}, {2, 5}

No further refinement is possible. Let:

A = {0, 3, 4}

B = {1}

C = {2, 5}

Reduced machine *M'*:

Present State	Next State Present Input 0	1	Output
A	C	A	1
B	C	C	0
C	B	A	0

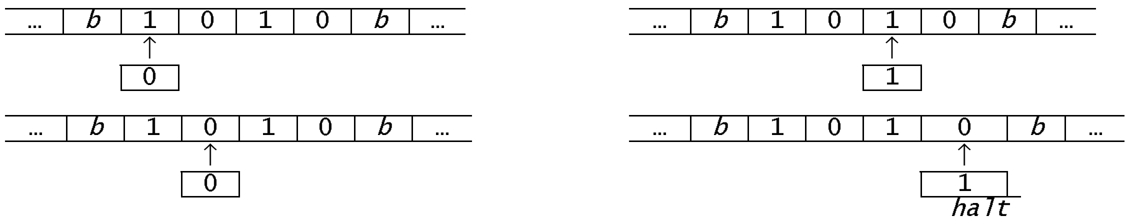
Section 8.3

2. Given the Turing machine:

(0, 1, 1, 0, *R*)
(0, 0, 0, 1, *R*)
(1, 1, 1, 1, *R*)
(1, *b*, 1, 2, *L*)
(2, 1, 1, 2, *L*)
(2, 0, 0, 2, *L*)
(2, *b*, 1, 0, *R*)

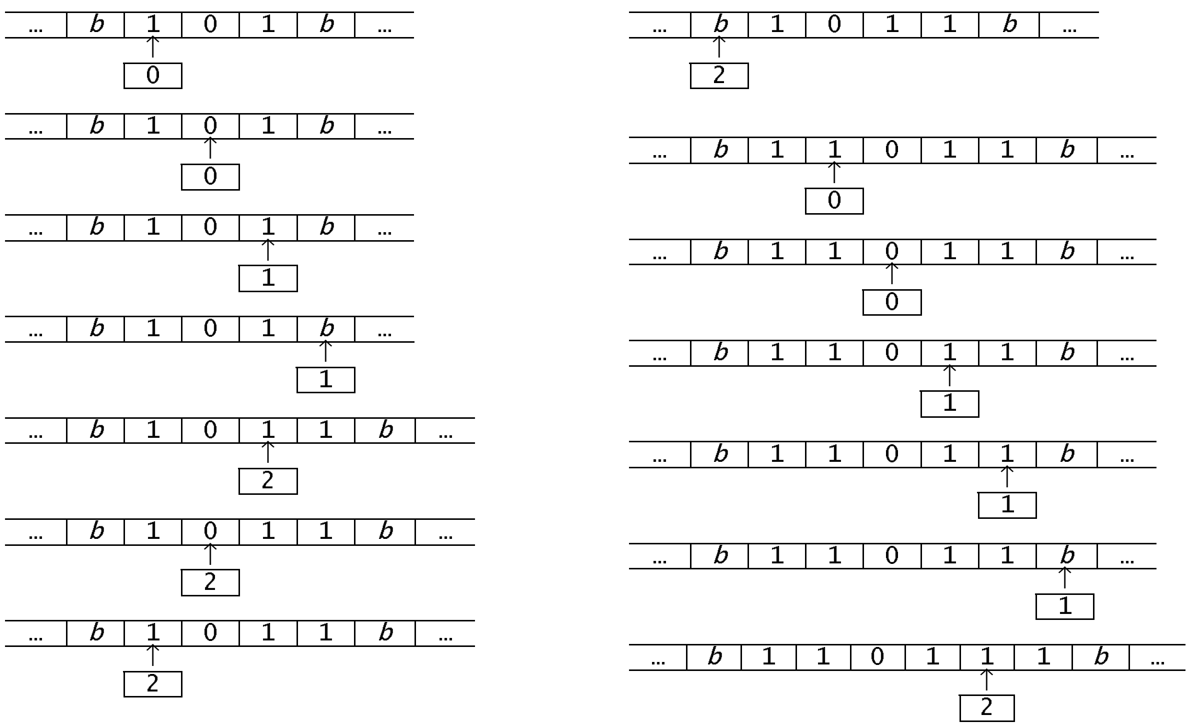
a. What is its behavior when started on the tape:

...	*b*	1	0	1	0	*b*	...



b. What is its behavior when started on the tape:

...	*b*	1	0	1	*b*	...



…and so on. The Turning machine does not recognize the pattern 101.

4. Find a Turing machine that recognizes the set of all 0s and 1s containing at least one 1.

(0, 0, 0, 0, *R*)
(0, 1, 1, 1, *R*)
(1, 0, 0, 1, *R*)
(1, 1, 1, 1, *R*)
(1, *b*, *b*, 2, *R*)

13. Find a Turing machine that, given an initial tape containing a nonempty string of 1s, marks the right end of the string with a * and puts a copy of th estring to the right of the *. As an example, the machine should, when started on a tape containing:

...	*b*	1	1	1	*b*	...

halt on a tape containing:

...	*b*	1	1	1	*	1	1	1	*b*	...

(0, 1, 1, 0, *R*)
(0, *b*, *, 1, *L*)

(1, 1, α, 2, *R*)
(1, *, *, 1, *L*)
(1, α, α, 1, *L*)
(1, *b*, *b*, 3, *R*)

(2, α, α, 2, *R*)
(2, *, *, 2, *R*)
(2, *b*, α, 1, *L*)

(3, α, 1, 3, *R*)
(3, *, *, 3, *R*)
(3, *b*, *b*, 4, *R*)

Section 8.4

7. Find a grammar that generates the set of all strings of well-balanced parentheses.

G = (V, V_T, S, P), where

V = { (,), S, A }
V_T = { (,) }

P consists of:

S → A
A → ()
A → (A)
A → AA

13. Find a context-free grammar that generates the language *L* where *L* consists of the set of all nonempty strings of 0s and 1s with twice as many 0s as 1s.

G = (V, V_T, S, P), where

V = { 0, 1, S, A, B }
V_T = { 0, 1 }

P consists of:

S → BBA
S → BAB
S → ABB
A → 1
A → 1S
B → 0
B → 0S

20a. The following is the *pumping lemma* for context-free languages. Let L be any context-free language. Then there exists some constant k such that for any word w in L with $|w| \geq k$, w can be written as the string $w_1w_2w_3w_4w_5$ with $|w_2w_3w_4| \leq k$ and $|w_2w_4| \geq 1$. Furthermore, the word $w_1w^i_2w_3w^i_4w_5 \in L$ for each $i \geq 0$.

Use the pumping lemma to show that $L = \{a^nb^nc^n \mid n \geq 1\}$ is not context free.

1.	Let L = $\{a^nb^nc^n \mid n \geq 1\}$	
2.	Assume L is context-free	hyp
3.	Pick an arbitrary integer constant k	
4.	Pick a word w from L such that $n > k$	
5.	$w = w_1w_2w_3w_4w_5$	pumping lemma, 2, 4
6.	$\|w_2\ w_3\ w_4\| \leq k$	pumping lemma, 2, 5
7.	$\|w_2\ w_4\| \geq 1$	pumping lemma, 2, 5
8.	for any word in L, there are n b's between the last a and 1st c	(1)
9.	$w_2w_3w_4$ can match $a+$, $a+b+$, $b+$, $b+c+$, or $c+$, but not $a+b+c+$	(4), (6), (8)
10.	$w_2\ w_3\ w_4$ cannot have both a's and c's	(9)
11.	$w_2\ w_4$ cannot have both a's and c's	(10)
12.	Pick an arbitrary integer i where $i > 1$	
13.	Let $w' = w_1\ w^i_2\ w_3\ w^i_4\ w_5$	
14.	$w' \in L$	pumping lemma
15.	$\|w'\| > \|w\|$	(7), (12), (13)
16.	w' couldn't have added equal numbers of a's and c's to w	(11), (13)
17.	w' doesn't have an equal number of a's and c's	(16)
18.	$w' \notin L$	(17), (1)
19.	contradiction	(14), (17)
20.	$\therefore L = \{a^nb^nc^n \mid n \geq 1\}$ is not context free	